Estimates of inverses of multivariable Toeplitz matrices

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GOHBERG-SEMENCUL

Let $P(z)=\sum_{i=0}^{n} P_i z^i$, $R(z)=\sum_{i=0}^{n} R_i(z)$ be invertible for $|z|\le 1$

and suppose that $P(z)P(z)^* = R(z)^* R(z)$. Put

$C(z) = P(z)^{*-1} P(z)^{-1} = R(z)^{-1} R(z)^{*-1} = \sum_{i=-\infty}^{\infty} C_i z^i$, $|z|=1$,

then

$$\begin{pmatrix} C_0 & \cdots & C_{-n+1} \\ \vdots & \ddots & \vdots \\ C_{n-1} & \cdots & C_0 \end{pmatrix}^{-1} = \begin{pmatrix} P_0 & & 0 \\ \vdots & \ddots & \\ P_{n-1} & \cdots & P_0 \end{pmatrix} \begin{pmatrix} P_0^* & \cdots & P_{n-1}^* \\ & \ddots & \vdots \\ 0 & & P_0^* \end{pmatrix} - \begin{pmatrix} R_n^* & & 0 \\ \vdots & \ddots & \\ R_1^* & \cdots & R_n^* \end{pmatrix} \begin{pmatrix} R_n & \cdots & R_1 \\ & \ddots & \vdots \\ 0 & & R_n \end{pmatrix}$$

[Gohberg-Heinig, 1974]

In scalar case $P(z)=R(z)$. [Gohberg-Semencul, 1972]

USEFUL LEMMA

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & * \\ * & * \end{pmatrix}$$ when inverses exist

<u>Corollary</u>: if

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}^{-1} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} P_{11}^* & P_{21}^* \\ 0 & P_{22}^* \end{pmatrix} = \begin{pmatrix} R_{11}^* & R_{21}^* \\ 0 & R_{22}^* \end{pmatrix} \begin{pmatrix} R_{11} & 0 \\ R_{21} & R_{22} \end{pmatrix}$$

with R_{22} invertible, then

$$A^{-1} = P_{11} P_{11}^* - R_{21}^* R_{21}$$

Note: only P_{11} and R_{21} have to be "nice", to get a "nice" formula.

PROOF OF G-S FORMULA

$C(z)^{-1} = P(z)P(z)^* = R(z)^*R(z)$ (with P, R stable) gives

$$\left[\left(C_{i-j}\right)_{i,j=0}^{\infty}\right]^{-1} = \begin{pmatrix} P_0 & & 0 & & 0 \\ \vdots & \ddots & & & \\ P_{n-1} & \cdots & P_0 & & \\ P_n & & & \ddots & \\ & \ddots & & & \\ 0 & & \ddots & & \end{pmatrix} \begin{pmatrix} P_0^* & \cdots & P_{n-1}^* & P_n^* & & 0 \\ & \ddots & \vdots & & \ddots & \\ & & P_0^* & & & \ddots \\ 0 & & & \ddots & & \\ & & & & \ddots & \end{pmatrix}$$

$$= \begin{pmatrix} \text{MESS} & R_n^* & & 0 \\ & \vdots & \ddots & \\ & R_1^* & & \ddots \\ & R_0^* & \ddots & \\ 0 & 0 & & \ddots \end{pmatrix} \begin{pmatrix} \text{MESS}^* & & & 0 \\ R_n & \cdots & R_1 & R_0 \\ & \ddots & & \ddots \\ 0 & & \ddots & \end{pmatrix}$$

Apply Corollary (and get rid of ∞ zero) $\square$

CRUCIAL OBSERVATION

$$\Pi_+\left[(P_0+\dots+P_n z^n)\,\Pi_+\left((P_0^*+\dots+P_n^* z^{-n})(z^n H_n + z^{n+1} H_{n+1} + \dots)\right)\right] =$$
$$\Pi_+\left[(R_0^*+\dots+R_n^* z^{-n})\,\Pi_+\left((R_0+\dots+R_n z^n)(z^n H_n + z^{n+1} H_{n+1} + \dots)\right)\right].$$

In d variables with $P(z)$, $R(z)$ stable of degree $n=(n_1,\dots,n_d)$; denote $z^n = (z_1,\dots,z_d)^{(n_1,\dots,n_d)} := z_1^{n_1}\cdots z_d^{n_d}$

Then

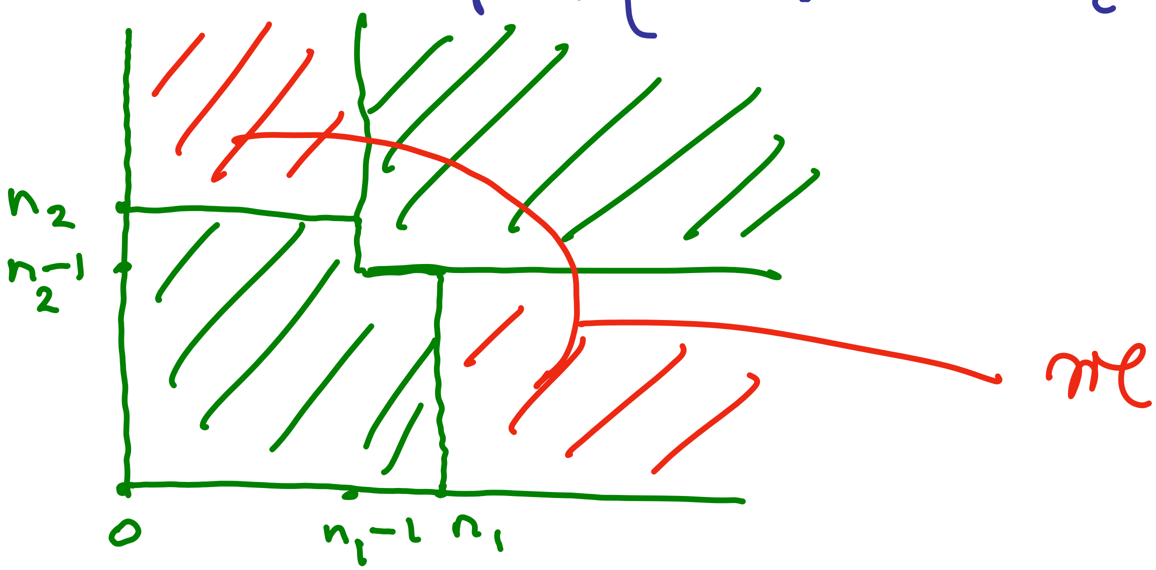
$$T_P T_P^* \Big|_{z^n H_2(\mathbb{T}^d)} = T_R^* T_R \Big|_{z^n H_2(\mathbb{T}^d)}$$

MAJOR DIFFERENCE

$d=1$: $H_2(\mathbb{T}) = \operatorname{span}\{1, \dots, z^{n-1}\} \oplus z^n H_2(\mathbb{T})$

$d>1$: $H_2(\mathbb{T}^d) = \operatorname{span}\{z^k \,;\, \underset{k \neq n}{k_i \leq n_i \ \forall i}\} \oplus \mathcal{M} \oplus z^n H_2(\mathbb{T}^d)$

where $\mathcal{M} = \operatorname{span}\{z^k : \exists i : k_i \leq n_i \ \& \ \exists j : k_j > n_j\}$



NEW LEMMA

if $\left[\left(A_{ij}\right)_{i,j=1}^{3}\right]^{-1} = \begin{pmatrix} P_{11} & & \\ P_{21} & P_{22} & \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \begin{pmatrix} P_{11}^* & P_{21}^* & P_{31}^* \\ & P_{22}^* & P_{32}^* \\ & & P_{33}^* \end{pmatrix}$

$$= \begin{pmatrix} R_{11}^* & R_{21}^* & R_{31}^* \\ & R_{22}^* & R_{32}^* \\ & & R_{33}^* \end{pmatrix} \begin{pmatrix} R_{11} & & \\ R_{21} & R_{22} & \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

with R_{22} and R_{33} invertible, then

$$A_{11}^{-1} = P_{11}P_{11}^* - R_{21}^*R_{21} - R_{31}^*R_{31} \leq P_{11}P_{11}^* - R_{31}^*R_{31}$$

previous Lemma

Loewner order

$\geq 0.$

THEOREM [W]

Let $P(z), R(z)$ be polynomials in $z=(z_1,\dots,z_d)$ of degree $n=(n_1,\dots,n_d)$ invertible on $\overline{\mathbb{D}}^d$ so that

$$Q(z) := P(z)P(z)^* = R(z)^* R(z), \quad z \in \mathbb{T}^d.$$

write

$$Q(z)^{-1} = \sum_{k \in \mathbb{Z}^d} c_k z^k, \quad z \in \mathbb{T}^d,$$

and let

$$\Lambda = \{k=(k_1,\dots,k_d) : 0 \le k_i \le n_i \ \forall i \ \& \ k \ne n\} \subset \mathbb{Z}^d.$$

Then

$$0 \le \left((c_{k-\ell})_{k,\ell\in\Lambda} \right)^{-1} \le AA^* - B^*B, \text{ where}$$

$$A = (P_{k-\ell})_{k,\ell\in\Lambda}, \quad B = (R_{k-\ell})_{\substack{k\in n+\Lambda \\ \ell\in\Lambda}}.$$

EXAMPLE

$d=2$, $n_1=n_2=2$

A

B

$$
\left(\begin{array}{cccccccc}
P_{00} & & & & & & & \\
P_{01} & P_{00} & & & & & & \\
P_{02} & P_{01} & P_{02} & & & & & \\
P_{10} & & & P_{00} & & & & \\
P_{11} & P_{10} & & P_{01} & P_{00} & & & \\
P_{12} & P_{11} & P_{10} & P_{02} & P_{01} & P_{00} & & \\
P_{20} & & & P_{10} & & & P_{00} & \\
P_{21} & P_{22} & & P_{11} & P_{10} & & P_{01} & P_{00}
\end{array}\right),
\left(\begin{array}{cccccccc}
R_{22} & R_{21} & R_{20} & R_{12} & R_{11} & R_{10} & R_{02} & R_{01} \\
 & R_{22} & R_{21} & & R_{12} & R_{11} & & R_{02} \\
 & & R_{22} & & & R_{12} & & \\
 & & & R_{22} & R_{21} & R_{20} & R_{12} & R_{11} \\
 & & & & R_{22} & R_{21} & & R_{12} \\
 & & & & & R_{22} & & \\
 & & & & & & R_{22} & R_{21} \\
 & & & & & & & R_{22}
\end{array}\right)
$$

HOW BIG IS DIFFERENCE

Examples:

d	n	size T	$\|T^{-1} - AA^* + B^*B\| / \|T^{-1}\|$
2	(5,5)	24	0.1415
2	(10,10)	99	0.2284
2	(20,20)	399	0.2403
2	(40,40)	1599	0.1968
3	(3,3,3)	26	0.1151
3	(5,5,5)	124	0.1755

USEFUL APPROXIMATION?

Use $X_0 := AA^* - B^*B$ as initial value for iterative scheme $X_k \longrightarrow T^{-1}$?

[Olshevsky, Oseledets, Tyrtyshnikov, 2006] use the Hotelling algorithm:

$$X_{n+1} = X_n(2I - TX_n),$$

which converges when

$$\| I - TX_0 \| < 1.$$

HOW TO GO : Toeplitz → P,R?

We need a way to go from $(c_{k-l})_{k,l\in\Lambda}$ to $P(z), R(z)$

In the scalar case $P(z) = R(z)$.

In general stable P, R do <u>not</u> exist so that

$$\widehat{P^{*-1}P^{-1}}(k) = \widehat{R^{-1}R^{*-1}}(k) = c_k;$$

in [Geronimo-W, 2004 & 2005] there are necessary and sufficient conditions when existence is assured.

"Idea": if $q(z) := \sum_{k\in n-n} c_k z^k > 0$ on $\mathbb{T}^d$, use Fourier coefficients of $\sqrt{\frac{1}{q}}$ instead (or $\sqrt{\frac{1}{q+\varepsilon}}$).

NUMERICAL RESULTS [with Gao & Koyuncu]

Consider $c(z_1, z_2) = 2 - \frac{1}{2}\left(z_1 + \frac{1}{z_1}\right) - \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$.

n_1	n_2	$\lVert I - TX_k \rVert$	k = #iter	cputime (in sec)	sizeT
16	16	5.9163e-09	8	0.37	256
32	32	1.6593e-09	10	8.72	1024
64	64	8.5383e-10	12	468.45	4096
128	128	6.1079e-10	14	21943.00	12384

For comparison, from [Olshevsky, Oseledets, Tyrtyshnikov, 2006]

128	128	5.5e-06	44	132.7	12384

3D EXAMPLE

$$t(z_1, z_2, z_3) = 6 + \left(z + \frac{1}{z_1}\right) + \left(z_2 + \frac{1}{z_2}\right) + \left(z_3 + \frac{1}{z_3}\right)$$

n_1	n_2	n_3	$\|I - TX_k\|$	k	cputime	size
8	8	8	1.7299e-15	6	1.51	512
16	16	16	1.4044e-12	6	272.32	4096