Preconditioners for ill conditioned (block) Toeplitz systems: facts and ideas

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Joint work with
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Problem

We are interested in the fast and efficient solution of $nm \times nm$ BTTB systems

$$T_{n,m}(f)x = b$$

where f is nonnegative real-valued belonging to $\mathcal{C}_{2\pi,2\pi}$ defined in the fundamental domain $Q = (-\pi, \pi]^2$ and is a priori known. The entries of the coefficient matrix are given by

$$t_{j,k} = \frac{1}{4\pi^2} \int_Q f(x,y) e^{-\mathbf{i}(jx+ky)} dxdy,$$

for $j = 0, \pm 1, \dots, \pm(n-1)$ and $k = 0, \dots, \pm(m-1)$.

Connection between f and $T_{nm}(f)$

The main connection between $T_{nm}(f)$ and the generating function is described by the the following result:

Theorem

If $f \in C[-\pi, \pi]^2$ and $c < C$ the extreme values of $f(x, y)$ on Q. Then every eigenvalue λ of the block Toeplitz matrix T satisfies the strict inequalities

$$c < \lambda < C$$

Moreover as $m, n \to \infty$ then

$$\lambda_{\min}(T_{nm}(f)) \to c \quad \text{and} \quad \lambda_{\max}(T_{nm}(f)) \to C$$

Crucial relationship

The basis for the construction of effective preconditioners is described by the following theorem

Theorem

Let $f, g \geq 0 \in C[-\pi, \pi]^2$ (f and g not identically zero). Then for every m, n the matrix $T_{nm}^{-1}(g)T_{nm}(f)$ has eigenvalues in the open interval (r, R), where

$$r = \inf_Q \frac{f}{g} \quad \textit{and} \quad R = \sup_Q \frac{f}{g}$$

Negative result for τ algebra preconditioners

Theorem (Noutsos,Serra,Vassalos, TCS (2004))

Let f be equivalent to $p_k(x,y) = (2-2\cos(x))^k + (2-2\cos(y))^k$ with $k \geq 2$ and let β be a fixed positive number independent of n. Then for every sequence $\{P_n\}$ with $P_n \in \tau$, $n = (n_1, n_2)$, and such that

$$\lambda_{\max}(P_n^{-1} T_n(f)) \leq \beta \tag{1}$$

uniformly with respect to $\hat{n}$, we have

(a) *the minimal eigenvalue of $P_n^{-1} T_n(f)$ tends to zero.*

(b) *the number*

$$\#\{\lambda(n) \in \sigma(P_n^{-1} T_n(f)) : \ \lambda(n) \to_{N(n)\to\infty} 0\}$$

tends to infinity as $N(\hat{n})$ tends to infinity.

Negative result for τ algebra preconditioners

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Let f be equivalent to $p_k(x,y) = (2-2\cos(x))^k + (2-2\cos(y))^k$ with $k \geq 2$ and let α be a fixed positive number independent of $n = (n_1, n_2)$. Then for every sequence $\{P_n\}$ with $P_n \in \tau$ and such that

$$\lambda_{\min}(P_n^{-1} T_n(f)) \geq \alpha \qquad (2)$$

uniformly with respect to n, we have

(a) *the maximal eigenvalue of $P_n^{-1} T_n(f)$ tends to ∞.*

(b) *the number*

$$\#\{\lambda(n) \in \sigma(P_n^{-1} T_n(f)) : \ \lambda(n) \to_{N(n)\to\infty} \infty\}$$

tends to infinity as $N(n)$ tends to infinity.

How to solve: 2D- ill conditioned problem

- Direct methods: Levinson type methods cost $O(n^2 m^3)$ ops while superfast methods $O(nm^3 \log^2 n)$. Stability problems. Not optimal.
- PCG method with matrix algebra preconditioners. Cost $O(k(\epsilon) nm \log nm)$, where $k(\epsilon)$ is the required number of iterations and depends from the condition number of T_{nm}.
- PCG method where the preconditioner is band Toeplitz matrix. Under some assumptions, the cost is optimal ($O(nm \log nm)$).
- Multigrid methods: Promising but in early stages. Cost $O(nm \log nm)$ ops. G. Fiorentino and S. Serra-Capizzano (1996), T. Huckle and J. Staudacher (2002),H.W. Sun, X.Q. Jin and Q.S. Chang (2004), A. Arico, M. Donatelli and S. Serra-Capizzano (2004).

More on Band Preconditioners

- S. Serra-Capizzano (BIT, (1994)) and M. Ng (LAA, (1997)) proposed as preconditioner the band BTTB matrix generated by the minimum trigonometric polynomial g which has the same roots with f.
- Let $f = g \cdot h$ with $h > 0$. Then D. Noutsos, S. Serra Capizzano and P. Vassalos (Numer. Math. (2006)) proposed as preconditioners the band BTTB matrix generated by $g \cdot \hat{h}$ where $\hat{h}$:
  1. is the trigonometric polynomial arises from the Fourier approximation on h.
  2. arises from the Lagrange interpolation of h at $2D$ Chebyshev points, or from the interpolation of h using the $2D$ Fejer kernel.

Preliminaries

We assume that the nonnegative function f has isolated zeros (x_1, y_1),(x_2, y_2),...,(x_k, y_k), on Q each one of multiplicities $(2\mu_1, 2\nu_1)$,$(2\mu_2, 2\nu_2)$,..., $(2\mu_k, 2\nu_k)$. Then, f can be written as

$$f = g \cdot w$$

where

$$g = \prod_{i=1}^{k} [(2 - 2\cos(x - x_i))^{\mu_i} + (2 - 2\cos(y - y_i))^{\nu_i}]$$

and w, is strictly positive on Q.

New Proposal

For the system

$$T_{nm}(f)x = b$$

we define and propose as a preconditioner the product of matrices

$$K_{nm}(f) = \mathcal{A}_{nm}(\sqrt{w})T_{nm}(g)\mathcal{A}_{nm}(\sqrt{w}) = \mathcal{A}_{nm}(h)T_{nm}(g)\mathcal{A}_{nm}(h)$$

with $h = \sqrt{w}$, $\mathcal{A}_{nm} \in \{\tau, \mathcal{C}, \mathcal{H}\}$, where $\{\tau, \mathcal{C}, \mathcal{H}\}$ is the set of matrices belonging to Block τ, Block Circulant and Block Hartley algebra, respectively.

Construction of $2D$ algebras

$\mathcal{A}$	$v^{[n]}$	Q_n
τ	$v_i^{[n]} = \frac{\pi i}{n+1}$	$\sqrt{\frac{2}{n+1}} \left(\sin(j v_i^{[n]}) \right)_{i,j=1}^{n}$
$\mathcal{C}$	$v_i^{[n]} = \frac{2\pi i}{n}$	$F_n = \frac{1}{\sqrt{n}} \left(e^{\mathbf{i} j v_i^{[n]}} \right)$
$\mathcal{H}$	$v_i^{[n]} = \frac{2\pi i}{n}$	$Re(F_n) + Im(F_n)$

The matrices $\mathcal{C}(h), \tau(h), \mathcal{H}(h)$ can be written as

$$\mathcal{A}_{nm}(h) = Q_{nm} \cdot Diag\left(f(v^{[nm]}) \right) \cdot Q_{nm}^H$$

where

$$v^{[nm]} = v^{[n]} \times v^{[m]} \quad \text{and} \quad Q_{nm} = Q_n \bigotimes Q_m$$

Obviously $K_{nm}(f)$ has all the properties that a preconditioner must satisfied, i.e,

- is symmetric,
- is positive definite,
- The cost for the solution of the arbitrary system

$$K_{nm}(f)x = b$$

is of order $O(nm \log nm)$. $O(nm \log nm)$ for the "inversion" of $\mathcal{A}_{nm}(h)$ by 2D FFT, and $O(nm)$ for the "inversion" of block band Toeplitz matrix $T_{n,m}(g)$ which can be done by multigrid methods. So, the only condition that must be fulfill, in order to be a competitive preconditioner, is the spectrum of $\left[K_{nm}^{\mathcal{A}}(f)\right]^{-1} T_{nm}(f)$ being bounded from above and below.

Useful Definition

Definition

We say that a function h is a $((k_1, k_2), (x_0, y_0))$-smooth function if

- $$\frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial y^{l_2}} h(x_0, y_0) = 0, \; l_1 < k_1, \;\; l_2 < k_2, \text{ and } l_1 + l_2 < \max\{k_1, k_2\}$$

- *and*
$$\frac{\partial^{l_1+l_2}}{\partial x^{l_1} \partial y^{l_2}} h(x_0, y_0)$$
is bounded for $l_1 = k_1, l_2 = 0$ *and* $l_2 = k_2, l_1 = 0$, *and* $l_1 + l_2 = \max\{k_1, k_2\}, l_1 < k_1, l_2 < k_2$.

τ Case: Clustering

Theorem

Let f belongs to the Wiener class. Then for every ϵ the spectrum of

$$[K^{\tau}_{nm}(f)]^{-1}\, T_{nm}(f)$$

lies in $[1-\epsilon, 1+\epsilon]$, for n, m sufficient large, except of $O(m+n)$ outliers. Thus, we have weak clustering around unity of the spectrum of the preconditioned matrix.

τ Case: Bounds

Theorem

Let $f \in C_{2\pi,2\pi}$ even function on Q with roots (x_1, y_1),(x_2, y_2), $\ldots$,(x_k, y_k), each one of multiplicities $(2\mu_1, 2\nu_1)$, $(2\mu_2, 2\nu_2)$, $\ldots$, $(2\mu_k, 2\nu_k)$, respectively. If g is the trigonometric polynomial of minimal degree that rises the roots of f and h is $(\mu_i - 1, \nu_i - 1)$ smooth function at the roots (x_i, y_i), $i = 1(1)k$, then the spectrum of the preconditioned matrix $P^\tau = [K^\tau_{nm}(f)]^{-1}\, T_{nm}(f)$ is bounded from above as well as bellow:

$$c < \lambda_{\min}(P^\tau) < \lambda_{\max}(P^\tau) < C$$

with c, C constants independent of n, m

Circulant Case: Clustering

Theorem

Let f belongs to the Wiener class on Q. Then for every ϵ the spectrum of

$$\left[K_{nm}^{\mathcal{C}}(f)\right]^{-1} T_{nm}(f)$$

lies in $[1-\epsilon, 1+\epsilon]$ for n, m sufficient large except $O(m+n)$ outliers. Thus, we have a weak clustering of the spectrum of the preconditioned matrix around unity.

Circulant Case: Bound of Spectrum

Theorem

Let $f \in C_{2\pi,2\pi}$ *even function on* Q *with roots* (x_1, y_1),(x_2, y_2), $\ldots$,(x_k, y_k), *on* Q *each one of multiplicities* $(2\mu_1, 2\nu_1)$,$(2\mu_2, 2\nu_2)$,$\ldots$, $(2\mu_k, 2\nu_k)$ *respectively. If* g *is the trigonometric polynomial of minimal degree that rises the roots of* f *and* h *is* (μ_i, ν_i) *smooth function at the roots* (x_i, y_i), $i = 1(1)k$ *then the spectrum of the preconditioned matrix* $P^{\mathcal{C}} = \left[K^{\mathcal{C}}_{nm}(f)\right]^{-1} T_{nm}(f)$ *is bounded above as well as bellow:*

$$c < \lambda_{\min}(P^{\mathcal{C}}) < \lambda_{\max}(P^{\mathcal{C}}) < C$$

with c, C *constants independent of* n, m

Case where h is not smooth enough

Let us suppose that f has roots at $(x_i, y_i), \quad i = 1(1)k$. If $h = \frac{f}{g}$ is not as smooth as the previous Theorems demand, then the spectrum of the preconditioned matrix could be unbound. For that case we make a “smoothing correction” and instead of h we use $\hat{h}$ defined as

$$\hat{h} = \begin{cases} h(x) & (x, y) \in Q / \bigcup \Omega_i \\ h(x_i, y_i) + \alpha_i g_i(x, y) & (x, y) \in \bigcup \Omega_i \end{cases}$$

where $\Omega_i = \{(x, y) : \|(x_i, y_i) - (x, y)\|_\infty < \epsilon_i\}$ and α_i is defined such that

$$h(\epsilon_i, \epsilon_i) = h(x_i, y_i) + \alpha_i g(\epsilon_i, \epsilon_i)$$

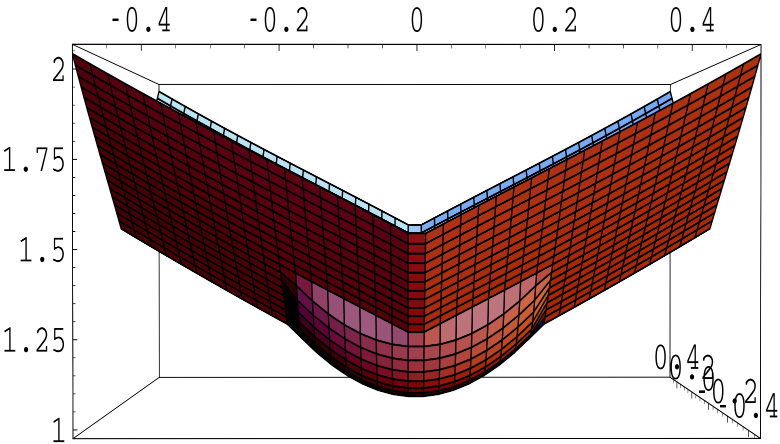

Figure: Smoothing of $h(x,y) = \frac{(|x|+|y|+1)(x^4+y^4)}{(2-2cos(x))^2+(2-2cos(y))^2}$, by $\hat{h}(x,y)$.

Numerical Experiments

We compare our proposal with the already known band preconditioners:

- B, which is generated by the trigonometric polynomial g that raises the roots of f
- K which is the product $g \cdot \hat{h}$ with $\hat{h}$ being the trigonometric polynomial arising from the 2D Fejer kernel on the positive part h of f
- P and F which arise: from the Lagrange interpolation of h at $2D$ Chebyshev points, and from the approximation of h using the $2D$ Fourier expansion, respectively.

For all the tests the stopping criterion was $\frac{\|r_k\|_2}{\|r_0\|_2} < 10^{-5}$, the starting vector the zero one and the righthand side vector of the system was $(1 \; 1 \; \cdots \; 1)^T$

Table: $f_1(x,y) = (x^4 + y^2)(|x| + |y|^3 + 1)$

$n = m$	B	$K_{4,4}$	$F_{4,4}$	$P_{4,4}$	τ	$\mathcal{C}$
8	17	12	11	9	8	10
16	47	21	15	15	10	14
32	73	28	20	18	12	17
64	91	31	22	20	13	19
128	103	33	23	21	13	21

Table: $f_2(x,y) = (x^2+y^2)(x^4+y^4+.1)$

$n=m$	B	$K_{6,6}$	$F_{6,6}$	$P_{6,6}$	τ	$\mathcal{C}$
8	10	11	-	10	7	9
16	48	17	18	27	10	14
32	81	21	98	80	12	18
64	94	23	*	*	13	21
128	*	24	*	*	14	23

*: The number of iterations exceeds 100

-: The matrix is singular.

Application to Toeplitz plus band systems

Let $f \in C_{2\pi,2\pi}$ even function on Q with roots (x_1, y_1),(x_2, y_2), $\ldots$,(x_k, y_k), on Q each one of multiplicities $(2\mu_1, 2\nu_1)$,$(2\mu_2, 2\nu_2)$,$\ldots$, $(2\mu_k, 2\nu_k)$ respectively. Consider the system

$$(T_{nm}(f) + B)x = b,$$

where B is a symmetric, p.d block band matrix. We can choose as preconditioner the

$$\mathcal{A}_{nm}(\sqrt{w})(T_{nm}(g) + B)\mathcal{A}_{nm}(\sqrt{w})$$

where $\{\tau, \mathcal{C}, \mathcal{H}\}$ is the set of matrices belonging to Block τ, Block Circulant and Block Hartley algebra, respectively, and $g(x, y)$ the trigonometric polynomial having the same roots with the same multiplicities with f.

Table: $f(x,y) = (|x||y|)(|x|+|y|+2)$

$n=m$	I	B	$\tau(f)$
8	10	10	5
16	50	20	6
32	174	39	7
64	603	62	8
128	*	98	9

We define as P_{nm} the matrix

$$Q_{nm} \cdot Diag\left(f(v^{[nm]})\right) \cdot Q_{nm}^H$$

where Q_{nm} is the orthogonal matrix diagonalize the τ algebra and $v_{ij}^{[nm]} = (\frac{\pi i}{n+1}, \frac{\pi j}{m+1})$. Then, the following theorem holds true:

Theorem

Let $f \in C_{2\pi,2\pi}$ even function on Q with root at $(0,0)$ with multiplicity (q,r) with $q,r \leq 2$. Then, the spectrum of $P_{nm}^{-1}T_{nm}(f)$ is clustered around unity. Moreover, for every n,m it holds that

$$c < \sigma(P_{nm}^{-1}T_{nm}(f)) < C$$

with $c, C > 0$ independent of n, m

Table: $f(x,y) = (|x| + |y|)(x^2 + y^2 + 1)$

$n = m$	$\#I$	$\lambda_{\min}(I)$	$\lambda_{\max}(P)$	$\lambda_{\min}(P)$	$\#P$
8	10	0.808	1.363	0.894	5
16	40	0.329	1.361	0.819	6
32	89	0.152	1.388	0.757	6
64	142	0.072	1.405	0.711	7
128	207	0.037	1.419	0.692	7

Thank you very much
for your attention!