# On tridiagonal matrices unitary equivalent with normal matrices

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# Outline

# Householder equivalence tridiagonalization

Given $A \in \mathbb{C}^{n\times n}$, $U_k$ and $V_k$ Householder transformations:

$$U_k^H \mathbf{x} = \omega \|\mathbf{x}\| \mathbf{e}_1, |\omega| = 1, \quad \text{and} \quad V_k^H \mathbf{y} = \sigma \|\mathbf{y}\| \mathbf{e}_1, |\sigma| = 1.$$

## Algorithm (Householder equivalence tridiagonalization)

*The algorithm computes $U^H AV = T$, with $T$ tridiagonal, $U$ and $V$ unitary.*

**For** *k=1:n-2*

*Compute the Householder reflector $U_k = I - \alpha \mathbf{v}\mathbf{v}^H$, based on $A(k+1:n,k)$*

$A(k+1:n,k:n) = U_k^H A(k+1:n,k:n)$

*Compute the Householder reflector $V_k = I - \beta \mathbf{w}\mathbf{w}^H$, based on $A(k,k+1:n)^H$*

$A(k:n,k+1:n) = A(k:n,k+1:n)V_k$

**end**

# Lanczos equivalence tridiagonalization

Suppose $U^H AV = T$, having diagonal elements $\alpha_i$, subdiagonals $\beta_i$ and superdiagonals $\gamma_i$ and $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ and $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$.
Based on

$$AV = UT \qquad \text{and} \qquad A^H U = VT^H$$

we get

$$\begin{aligned} A\mathbf{v}_k &= \gamma_{k-1}\mathbf{u}_{k-1} + \alpha_k \mathbf{u}_k + \beta_k \mathbf{u}_{k+1} \quad (1) \\ A^H \mathbf{u}_k &= \overline{\beta}_{k-1}\mathbf{v}_{k-1} + \overline{\alpha}_k \mathbf{v}_k + \overline{\gamma}_k \mathbf{v}_{k+1}, \quad (2) \end{aligned}$$

Rewriting (1) and (2) gives us (with $\alpha_k = \mathbf{u}_k^H A\mathbf{v}_k = \overline{\mathbf{v}_k^H A^H \mathbf{u}_k}$):

$$\begin{aligned} \mathbf{r}_{k+1} &= A\mathbf{v}_k - \gamma_{k-1}\mathbf{u}_{k-1} - \alpha_k \mathbf{u}_k, \\ \mathbf{s}_{k+1} &= A^H \mathbf{u}_k - \overline{\beta}_{k-1}\mathbf{v}_{k-1} - \overline{\alpha}_k \mathbf{v}_k. \end{aligned}$$

Hence $\beta_k = \|\mathbf{r}_{k+1}\|_2$, $\mathbf{u}_{k+1} = \mathbf{r}_{k+1}/\beta_k$ and $\gamma_k = \|\mathbf{s}_{k+1}\|_2$, $\mathbf{v}_{k+1} = \mathbf{s}_{k+1}/\gamma_k$.

# Lanczos equivalence Tridiagonalization

## Algorithm (Lanczos equivalence tridiagonalization)

*The algorithm computes "theoretically" $U^H AV = T$, with $T$ tridiagonal, $U$ and $V$ unitary.*

*Initialize* $\mathbf{u}_1$ *and* $\mathbf{v}_1$. *(E.g.,* $\mathbf{u}_1 = \mathbf{e}_1 = \mathbf{v}_1$.*)*

***for*** $k = 1 : n-1$

$$\alpha_k = \mathbf{u}_k^H A\mathbf{v}_k$$

$$\mathbf{r} = A\mathbf{v}_k - \gamma_{k-1}\mathbf{u}_{k-1} - \alpha_k \mathbf{u}_k$$

$$\mathbf{s} = A^H\mathbf{u}_k - \overline{\beta}_{k-1}\mathbf{v}_{k-1} - \overline{\alpha}_k \mathbf{v}_k$$

$$\beta_k = \omega\|\mathbf{r}\|_2, \quad \gamma_k = \sigma\|\mathbf{s}\|_2 \qquad (\omega, \sigma \text{ are free, } |\omega| = |\sigma| = 1)$$

$$\mathbf{u}_{k+1} = \mathbf{r}/\beta_k, \quad \mathbf{v}_{k+1} = \mathbf{s}/\gamma_k$$

***end***

# Essential uniqueness: Case 1

Case 1: sub- and superdiagonal elements different from zero.

## Theorem

*$A \in \mathbb{C}^{n\times n}$, $U, V$ unitary, $T, S$ tridiagonal:*

$$T = U^H A V, \qquad S = \hat{U}^H A \hat{V}.$$

*sub- and superdiagonal elements different from zero.*

*When*

$$U\mathbf{e}_1 = \hat{\omega}\hat{U}\mathbf{e}_1, \quad V\mathbf{e}_1 = \omega\hat{V}\mathbf{e}_2, \quad |\omega_1| = |\hat{\omega}_1| = 1.$$

*then unitary diagonal $D$ and $\hat{D}$ exist, such that*

$$VD = \hat{V}, \quad U\hat{D} = \hat{U} \quad \text{and} \quad |T| = |S|.$$

# Essential uniqueness: Case 2

Case 2: sub- and superdiagonal elements can be zero.

## Theorem

*Same assumptions as before;*

$$K = \min\{i | s_{i+1,i} = 0\}, \quad \textit{and} \quad L = \min\{i | s_{i,i+1} = 0\}.$$

*Then we have three different cases:*

- $K < L$.
  - *Columns* $1$ *up to* $K$ *of* $U$ *and* $\hat{U}$ *are essentially unique.*
  - *Columns* $1$ *up to* $K+1$ *of* $V$ *and* $\hat{V}$ *are essentially unique.*
  - *For* $1 \leq k \leq K$ *and* $1 \leq l \leq K+1$: $|t_{k,l}| = |s_{k,l}|$.
- $L < K$. *Similar.*
- $K = L$. *Similar.*

# Essential uniqueness: Case 2

Below the resulting $T$ is depicted:
The $\boxtimes$ denote the essentially unique parts.

| $K < L$ and $K = 3$ | $K > L$ and $L = 3$ | $K = L = 3$ |
|---|---|---|
| $\begin{bmatrix} \boxtimes & \boxtimes & & & \\ \boxtimes & \boxtimes & \boxtimes & & \\ & \boxtimes & \boxtimes & \boxtimes & \\ & & 0 & \times & \times \\ & & & \times & \times \end{bmatrix}$ | $\begin{bmatrix} \boxtimes & \boxtimes & & & \\ \boxtimes & \boxtimes & \boxtimes & & \\ & \boxtimes & \boxtimes & 0 & \\ & & \boxtimes & \times & \times \\ & & & \times & \times \end{bmatrix}$ | $\begin{bmatrix} \boxtimes & \boxtimes & & & \\ \boxtimes & \boxtimes & \boxtimes & & \\ & \boxtimes & \boxtimes & 0 & \\ & & 0 & \times & \times \\ & & & \times & \times \end{bmatrix}$ |

# Outline

1. Unitary Equivalence relation
   - Householder equivalence tridiagonalization
   - Lanczos equivalence tridiagonalization
   - Essential uniqueness

2. **The normal case**
   - Main theorem
   - Scalar product spaces
   - Specific reductions
   - Few extra properties

3. Associated Krylov spaces
   - Krylov subspaces
   - Krylov matrices
   - Examples

4. Eigenvalues and singular values

5. Conclusions

# Main theorem

## Theorem

*Given a normal* $A \in \mathbb{C}^{n \times n}$.

*For* $U, V$, *with* $U\mathbf{e}_1 = \omega V \mathbf{e}_1$ ($|\omega| = 1$) *such that*

$$U^H A V = T$$

*with* $T$ *tridiagonal having*

$$\begin{array}{rl} \textit{subdiagonal elements} & \beta_i, \\ \textit{superdiagonal elements} & \gamma_i. \end{array}$$

*We have (assume* $\gamma_i$ *and* $\beta_i$ *different from* $0$*):*

$$|\beta_i| = |\gamma_i|, \quad \forall i = 1, \ldots, n-1.$$

In case a $\gamma_i$ and/or $\beta_i$ is zero, a sort of restart or equivalently and extra relation needs to be put on $U$ and $V$.

# Comments on the proof

By induction on $k$ (three steps):

1. $|\gamma_k| = |\beta_k|$.
2. A recurrence in bivariate polynomials is proven for $A^H\mathbf{u}_{k+1}$ and $A\mathbf{v}_{k+1}$:

$$\begin{aligned} A^H\mathbf{u}_{k+1} &= \frac{1}{\beta_{1:k}}\left(A^H \frac{\beta_{1:k-1}}{\bar{\gamma}_{1:k-1}}\,\bar{p}_k(A^H,A) - \beta_{k-1}\gamma_{k-1}p_{k-1}(A,A^H) - \alpha_k p_k(A,A^H)\right)\mathbf{v}_1 \\ &= \frac{1}{\beta_{1:k}} p_{k+1}(A,A^H)\mathbf{v}_1 \end{aligned}$$

   and a similar relation

$$A\mathbf{v}_{k+1} = \frac{1}{\bar{\gamma}_{1:k}}\bar{p}_{k+1}(A^H,A)\mathbf{v}_1,$$

   $\beta_0 = \gamma_0 = 0, p_0 = 0$ and $p_1(x,y) = y$.

3. Based on these results we get $\|A\mathbf{v}_{k+1}\|_2 = \|A^H\mathbf{u}_{k+1}\|_2$.

This has also consequences on the implementation.

# Scalar product spaces

- For $A$ normal we have a factorization

$$U^H AV = T = SD,$$

with $S$ complex symmetric and $D$ unitary diagonal.

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- Consider the bilinear form ($\Omega$ is a weight matrix):

$$\langle \mathbf{x}, \mathbf{y} \rangle_\Omega = \mathbf{x}^T \Omega \mathbf{y}.$$

The adjoint of $A$ w.r.t. $\langle \cdot, \cdot \rangle_\Omega$ is $A^\star$:

$$\langle A\mathbf{x}, \mathbf{y} \rangle_\Omega = \langle \mathbf{x}, A^\star \mathbf{y} \rangle_\Omega, \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{C}^n.$$

A closed formula:

$$A^\star = \Omega^{-1} A^T \Omega,$$

# Scalar product spaces

- For $A$ normal we have a factorization
$$U^H AV = T = SD,$$
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A closed formula:
$$A^\star = \Omega^{-1} A^T \Omega,$$

- It is easily checked that for $\Omega = D$:
$$\begin{aligned} T^\star &= D^{-1} T^T D, \\ &= D^{-1} (SD)^T D, \\ &= T. \end{aligned}$$
Hence, $T$ is self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_D$.

# Some corollaries

Compact formulation of the main theorem.

## Theorem

*For $A \in \mathbb{C}^{n \times n}$ normal and $U^H AV = T$, satisfying the conditions above we get: $T$ is self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_\Omega$, with $\Omega$ a unitary diagonal matrix.*

# Some corollaries

Compact formulation of the main theorem.

**Theorem**

*For $A \in \mathbb{C}^{n \times n}$ normal and $U^H AV = T$, satisfying the conditions above we get: $T$ is self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_\Omega$, with $\Omega$ a unitary diagonal matrix.*

We have an even stronger result.

**Theorem**

*For $A \in \mathbb{C}^{n \times n}$ normal and $\Omega$ a unitary diagonal.*
*There exists $U$ and $V$ ... such that $U^H AV = T$ is tridiagonal and $T$ is self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_\Omega$.*

# Specific reductions

Construct the unitary matrices $U$ and $V$ such that $U^H AV = T$.

- $T$ is tridiagonal having superdiagonals $\gamma_i$ and subdiagonals $\beta_i$:
  1. $\gamma_i = \beta_i$ Symmetric reduction.
  2. $\gamma_i = \pm\beta_i$ Pseudo-Symmetric reduction.
  3. $\gamma_i = -\beta_i$ Skew-Symmetric reduction.
  4. $\gamma_i = \overline{\beta}_i$ Hermitian reduction.
  5. $\gamma_i = \pm\overline{\beta}_i$ Pseudo-Hermitian reduction.
  6. $\gamma_i = -\overline{\beta}_i$ Skew-Hermitian reduction.

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- Examples of the nomenclature:
  (all related to certain $\langle\cdot,\cdot\rangle_\Omega$ [2×Mackey + Tisseur, SIMAX, 2005])
  1. Signature matrix: $D$ has $\pm 1$ on the diagonal.
  2. Pseudo-symmetric: $A = SD$, with $S$ symmetric, $D$ signature matrix.
  3. Complex pseudo skew-symmetric: $A = SD$, $S$ complex skew-symmetric, $D$ a signature matrix.

# Structure of the resulting tridiagonal matrix

Left: specific normal matrices.
Top: type of reduction performed, with relation between $\gamma_i$ and $\beta_i$.

| **Matrix** | $\mathbb{F}$ | **Arb.** $(\Omega)$ $\lvert\gamma_i\rvert = \lvert\beta_i\rvert$ | **Sym.** $(\Omega = I)$ $\gamma_i = \beta_i, \quad \gamma_i, \beta_i \in \mathbb{R}$ | **Pseu.-Sym.** $(\Omega = D)$ $\gamma_i = \pm\beta_i, \quad \gamma_i, \beta_i \in \mathbb{R}$ | **Sk.-Sym.** $(\Omega = \Sigma)$ $\gamma_i = -\beta_i, \quad \gamma_i, \beta_i \in \mathbb{R}$ |
|---|---|---|---|---|---|
| Normal | $\mathbb{R}$ | Pseu.-Sym. | Sym. | Pseu.-Sym. | Pseu.-Sym. |
| Sym. | $\mathbb{R}$ | Pseu.-Sym. | **Sym.** | Pseu.-Sym. | Pseu.-Sym. |
| Sk.-Sym. | $\mathbb{R}$ | Pseu.-Sk.-Sym. | Pseu.-Sk.-Sym. | Pseu.-Sk.-Sym. | **Sk.-Sym.** |
| Orth. | $\mathbb{R}$ | Pseu.-Sym.<br>Orth.<br>Block-Diag. | Sym.<br>Orth.<br>Block-Diag. | Pseu.-Sym.<br>Orth.<br>Block-Diag. | Pseu.-Sym<br>Orth.<br>Block-Diag. |
| Normal | $\mathbb{C}$ | X | Cplx.-Sym. | Cplx. Pseu.-Sym. | Cplx. Pseu.-Sym. |
| Herm. | $\mathbb{C}$ | X | Cplx.-Sym. | Cplx. Pseu.-Sym. | Cplx. Pseu.-Sym. |
| Sk.-Herm. | $\mathbb{C}$ | X | Cplx.-Sym. | Cplx. Pseu.-Sym. | Cplx. Pseu.-Sym. |
| Unitary | $\mathbb{C}$ | X<br>Unit.<br>Block-Diag. | Cplx.-Sym.<br>Unit.<br>Block-Diag. | Cplx. Pseu.-Sym.<br>Unit.<br>Block-Diag. | Cplx. Pseu.-Sym.<br>Unit.<br>Block-Diag. |

# Structure of the resulting tridiagonal matrix

| **Matrix Type** | $\mathbb{F}$ | **Herm.** $\gamma_i = \overline{\beta}_i, \quad \gamma_i, \beta_i \in \mathbb{C}$ | **Pseu.-Herm.** $\gamma_i = \pm\overline{\beta}_i, \quad \gamma_i, \beta_i \in \mathbb{C}$ | **Skew-Herm.** $\gamma_i = -\overline{\beta}_i, \quad \gamma_i, \beta_i \in \mathbb{C}$ |
|---|---|---|---|---|
| Normal | $\mathbb{R}$ | Sym. | Pseu.-Sym. | Pseu.-Sym. |
| Sym | $\mathbb{R}$ | Sym. | Pseu.-Sym. | Pseu.-Sym. |
| Skew-Sym. | $\mathbb{R}$ | Pseu.-Skew-Sym. | Pseu.-Skew-Sym. | Skew-Sym. |
| Orthogonal | $\mathbb{R}$ | Sym. Orth. Block-Diag. | Pseu.-Sym Orth. Block-Diag. | Pseu.-Sym Orth. Block-Diag. |
| Normal | $\mathbb{C}$ | X | X | X |
| Herm. | $\mathbb{C}$ | **Herm.** | Pseu.-Herm. | Pseu.-Herm. |
| Skew-Herm. | $\mathbb{C}$ | Pseu.-Skew-Herm. | Pseu.-Skew-Herm. | **Skew-Herm.** |
| Unitary | $\mathbb{C}$ | Unitary Block-Diag. | Unitary Block-Diag. | Unitary Block-Diag. |

# Link with well-known methods

One can easily prove that:

- Applying symmetric reduction on symmetric matrix
  $\longrightarrow$ standard similarity.
- Applying skew-symmetric reduction on skew-symmetric matrix
  $\longrightarrow$ standard similarity.
- Applying hermitian reduction on hermitian matrix
  $\longrightarrow$ standard similarity.
- Applying skew-hermitian reduction on skew-hermitian matrix
  $\longrightarrow$ standard similarity.

# Few extra properties

Suppose $U^H AV = T$, with $T$ complex symmetric (i.e. $\gamma_i = \beta_i$):

- $UV^H, VU^H$ are complex symmetric.
- $U^H A^i V$ (with $i \in \mathbb{Z}$) is complex symmetric.
- $V^H A^i U$ (with $i \in \mathbb{Z}$) is complex symmetric.
- $U^H (A^H)^i V$ (with $i \in \mathbb{Z}$) is complex symmetric.
- $V^H (A^H)^i U$ (with $i \in \mathbb{Z}$) is complex symmetric.
- $U^H p(A, A^H, A^{-1}) V$ is complex symmetric ($p$ a polynomial).
- $V^H p(A, A^H, A^{-1}) U$ is complex symmetric ($p$ a polynomial).
- $U^H AU = \overline{V^H A^H V}$.
- $A = (UT\overline{U})(U^T V^H)$ is a unitary complex symmetric factorization.

# Outline

# Krylov subspace approach

We switch back to the arbitrary matrix case:

**$A$ is not necessarily normal anymore!**

- Let us define the following 'cyclical' Krylov sequences ($k$ equals the number of terms):

$$\mathcal{C}_k(A,\mathbf{x}) = \text{span}\{\mathbf{x}, A\mathbf{y}, AA^H\mathbf{x}, AA^HA\mathbf{y}, (AA^H)^2\mathbf{x}, \ldots\}$$
$$\mathcal{C}_k(A^H,\mathbf{y}) = \text{span}\{\mathbf{y}, A^H\mathbf{x}, A^HA\mathbf{y}, A^HAA^H\mathbf{x}, (AA^H)^2\mathbf{y}, \ldots\}.$$

- NOTE: this is NOT the unsymmetric Lanczos process ($W^HU = I$)!

$$\mathcal{K}_k(A,\mathbf{u}_1) = \text{span}\{\mathbf{u}_1, A\mathbf{u}_1, A^2\mathbf{u}_1, A^3\mathbf{u}_1, \ldots\}$$
$$\mathcal{K}_k(A^H,\mathbf{w}_1) = \text{span}\{\mathbf{w}_1, A^H\mathbf{w}_1, (A^H)^2\mathbf{w}_1, (A^H)^3\mathbf{w}_1, \ldots\}.$$

# Krylov subspace approach

- Considering the following Krylov space

$$\mathcal{K}\left(\begin{bmatrix} 0 & A \\ A^H & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right)$$
$$= \operatorname{span}\left\{\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} A\mathbf{y} \\ A^H\mathbf{x} \end{bmatrix}, \begin{bmatrix} AA^H\mathbf{x} \\ A^HA\mathbf{y} \end{bmatrix}, \begin{bmatrix} AA^HA\mathbf{y} \\ A^HAA^H\mathbf{x} \end{bmatrix}, \ldots\right\}$$

- Similar results as for the cyclical Krylov approach can be obtained when putting also the vectors **x** and **y** in a matrix and apply block Lanczos.

- This leads to a sort of block product Krylov subspace process. [Kressner, Watkins, ...]

# Krylov subspace approach

- Let us define the following 'cyclical' Krylov sequences ($k$ equals the number of terms):

$$\begin{aligned} \mathcal{C}_k(A,\mathbf{x}) &= \text{span}\{\mathbf{x}, A\mathbf{y}, AA^H\mathbf{x}, AA^HA\mathbf{y}, (AA^H)^2\mathbf{x}, \ldots\} \\ \mathcal{C}_k(A^H,\mathbf{y}) &= \text{span}\{\mathbf{y}, A^H\mathbf{x}, A^HA\mathbf{y}, A^HAA^H\mathbf{x}, (AA^H)^2\mathbf{y}, \ldots\}. \end{aligned}$$

- We have

$$\begin{aligned} A\mathcal{C}_k(A^H,\mathbf{y}) &\subset \mathcal{C}_{k+1}(A,\mathbf{x}), \\ A^H\mathcal{C}_k(A,\mathbf{x}) &\subset \mathcal{C}_{k+1}(A,\mathbf{y}). \end{aligned}$$

- Consider two orthogonal bases ($\forall k$):

$$\begin{aligned} \{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3,\ldots,\mathbf{u}_k\} &\quad \text{spanning} \quad \mathcal{C}_k(A,\mathbf{x}), \\ \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\ldots,\mathbf{v}_k\} &\quad \text{spanning} \quad \mathcal{C}_k(A^H,\mathbf{y}). \end{aligned}$$

# Orthogonal relations

Since

$$\mathbf{v}_k \in \mathcal{C}_{k+1}(A^H,\mathbf{y})\backslash\mathcal{C}_k(A^H,\mathbf{y}) \quad \text{and} \quad \langle A\mathbf{v}_k,\mathbf{u}_i\rangle = \langle \mathbf{v}_k, A^H\mathbf{u}_i\rangle,$$
$$\mathbf{u}_k \in \mathcal{C}_{k+1}(A,\mathbf{x})\backslash\mathcal{C}_k(A,\mathbf{x}) \quad \text{and} \quad \langle A^H\mathbf{u}_k,\mathbf{v}_i\rangle = \langle \mathbf{u}_k, A\mathbf{v}_i\rangle.$$

we have (for $1 \leq i \leq k-2$),

$$A\mathbf{v}_k \perp \mathbf{u}_i \text{ and } A\mathbf{u}_k \perp \mathbf{v}_i.$$

Hence, we get (assume $\beta_i$ and $\gamma_i$ different from zero):

$$\begin{aligned} A\mathbf{v}_i &= \gamma_{i-1}\mathbf{u}_{i-1} + \alpha_i\mathbf{u}_i + \beta_{i+1}\mathbf{u}_{i+1}, \\ A^H\mathbf{u}_i &= \bar{\beta}_{i-1}\mathbf{v}_{i-1} + \bar{\alpha}_i\mathbf{v}_i + \bar{\gamma}_{i+1}\mathbf{v}_{i+1}, \end{aligned}$$

where $\beta_{i+1} = \langle \mathbf{u}_{i+1}, A\mathbf{v}_i\rangle, \alpha_i = \langle \mathbf{u}_i, A\mathbf{v}_i\rangle$ and $\gamma_{i-1} = \langle \mathbf{u}_{i-1}, A\mathbf{v}_i\rangle$.
This leads to

$$\begin{aligned} AV_k &= U_k T_k + \beta_{k+1}\mathbf{u}_{k+1}\mathbf{e}_k^T, \\ A^H U_k &= V_k T_k^H + \bar{\gamma}_{k+1}\mathbf{v}_{k+1}\mathbf{e}_k^T. \end{aligned}$$

# Cyclical Krylov matrices

Consider cyclical krylov matrices:

$$\begin{aligned} C_k(A,\mathbf{x}) &= \left[\mathbf{x}, A\mathbf{y}, AA^H\mathbf{x}, AA^HA\mathbf{y}, (AA^H)^2\mathbf{x}, \ldots\right] \\ C_k(A^H,\mathbf{y}) &= \left[\mathbf{y}, A^H\mathbf{x}, A^HA\mathbf{y}, A^HAA^H\mathbf{x}, (AA^H)^2\mathbf{y}, \ldots\right]. \end{aligned}$$

To prove the main theorem a small lemma is needed.

## Lemma

*Suppose $AV = U\hat{A}$ and $A^HU = V\hat{A}^H$ then:*

$$\begin{aligned} UC_k(\hat{A},\mathbf{x}) &= C_k(A, U\mathbf{x}), \\ VC_k(\hat{A}^H,\mathbf{y}) &= C_k(A, V\mathbf{y}). \end{aligned}$$

## Theorem

*For $U$ and $V$ unitary, $U^H AV = T$ is tridiagonal if and only if the columns of $U$ and $V$ define an orthonormal basis for a specific cyclical Krylov subspace.*

- The $\Leftarrow$ is proved before.
- The $\Rightarrow$: $T$ is tridiagonal, hence we have ($R, \hat{R}$ upper triangular):

$$C_k(T, \mathbf{e}_1) = R \text{ and } C_k(T^H, \mathbf{e}_1) = \hat{R};$$

Since $AV = UT$ and $A^H U = VT^H$ we can apply the lemma:

$$\begin{aligned} UR = UC_k(T, \mathbf{e}_1) &= C_k(A, \mathbf{u}_1), \\ V\hat{R} = VC_k(T^H, \mathbf{e}_1) &= C_k(A^H, \mathbf{v}_1). \end{aligned}$$

On the left two $QR$-factorizations are shown, hence $U$ and $V$ define an orthonormal basis for $\mathcal{C}_k(A, \mathbf{u}_1)$ and $\mathcal{C}_k(A^H, \mathbf{v}_1)$ respectively.

# (Skew-)Hermitian matrices

- When $A = A^H$ is Hermitian, we get:

$$\mathcal{C}_k(A^H, \mathbf{x}) = \mathcal{C}_k(A, \mathbf{x}) \quad = \quad \text{span}\{\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, \ldots, A^{k-1}\mathbf{x}\}.$$

  this is that standard Krylov subspace $\mathcal{K}_k(A, \mathbf{x})$.
- Hence we have $U^H A U = T$ with $U$ unitary, $T$ hermitian.

- When $-A = A^H$ is skew-Hermitian we get:

$$\begin{aligned}
\mathcal{C}_k(A, \mathbf{x}) &= \text{span}\{\mathbf{x}, A\mathbf{x}, -A^2\mathbf{x}, -A^3\mathbf{x}, A^4\mathbf{x}, \ldots\}, \\
\mathcal{C}_k(A^H, \mathbf{x}) &= \text{span}\{\mathbf{x}, -A\mathbf{x}, -A^2\mathbf{x}, A^3\mathbf{x}, A^4\mathbf{x}, \ldots\}, \\
\mathcal{K}_k(A, \mathbf{x}) &= \text{span}\{\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, A^3\mathbf{x}, A^4, \mathbf{x}, \ldots, \}.
\end{aligned}$$

- Hence we have $U^H A U = T$ with $U$ unitary, $T$ skew-hermitian.

# Unitary matrices

- $A$ is unitary $AA^H = A^H A = I$.
- Let us distinguish between two cases:
  - Starting vector is an eigenvector $\mathbf{v}$.

$$\begin{aligned} \mathcal{C}_2(A, \mathbf{v}) &= \mathcal{C}_1(A, \mathbf{v}) \\ \mathcal{C}_2(A^H, \mathbf{v}) &= \mathcal{C}_1(A^H, \mathbf{v}) \end{aligned}$$

    as a result we have a $1 \times 1$ block on the diagonal and a restart is required.
  - If $\mathbf{v}$ is not an eigenvector:

$$\begin{aligned} \mathcal{C}_3(A, \mathbf{v}) &= \text{span}\{\mathbf{v}, A\mathbf{v}, AA^H\mathbf{v}\} \\ &= \text{span}\{\mathbf{v}, A\mathbf{v}, I\mathbf{v}\} \\ &= \text{span}\{\mathbf{v}, A\mathbf{v}\} = \mathcal{C}_2(A, \mathbf{v}). \end{aligned}$$

    and also

$$\mathcal{C}_3(A^H, \mathbf{v}) = \mathcal{C}_2(A^H, \mathbf{v}),$$

    as a result we have a $2 \times 2$ block on the diagonal and a restart is required.
- So generically always a block tridiagonal with $2 \times 2$ blocks on the diagonal. (Eventually a trailing $1 \times 1$ block when $n$ is odd.)

# Outline

# General remarks

- Intimitely related:

$$
\begin{aligned}
A &= W\Delta W^H && \text{Eigenvalue decomposition,} \\
A &= U\Sigma V^H && \text{Singular value decomposition,} \\
\Sigma &= |\Delta| = \Delta D && D \text{ unitary diagonal.} \\
V^H U &= D.
\end{aligned}
$$

- Given the eigenvalues $\Delta$:
  $\rightarrow$ we have the singular values $\Sigma = |\Delta|$.
- Given the eigenvalue decomposition $W\Delta W^H$:
  $\rightarrow$ we have the SVD: $W\,|\Delta|\,(\overline{D}W^H) = W\,\Sigma(\overline{D}W^H)$ .
- Given the singular values $\Sigma$:
  $\rightarrow$ NOT possible to compute eigenvalues.
- Given the singular value decomposition $U\Sigma V^H$:
  $\rightarrow$ we have the eigenvalue decomposition since we have
  $V^H U = D$ and $\Delta = \Sigma\overline{D}$.

For the moment no efficient technique for computing the eigenvalues, exploiting the normal matrix structure exists.

# SVD-method for normal matrices

Computing the full SVD.

- Reduction to bidiagonal form $B$:
  $\rightarrow$ Compute and store $2n-3$ Householder transformations.
- Compute SVD of $B = U\Sigma V^H$:
  $\rightarrow$ combine all performed chasing transformations,
  $\rightarrow$ store the two unitary matrices $U$ and $V$.

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- Reduction to complex tridiagonal form $T$:
  $\rightarrow$ Compute and store $2n-4$ Householder transformations.
- Compute SVD of $T$:
  - $T$ is complex symmetric, hence SVD $\rightarrow$ Takagi factorization.

    $$T = U\Sigma V^H = Q\Sigma Q^T,$$

    with $Q$ unitary.
  - Based on the $QR$-iteration on $TT^H$.
  - Faster, less memory than the standard SVD, when $Q$ is desired.
  - See [Gragg, Bunse-Gerstner, JCAM, 1988]

# Outline

# Conclusions

- Tridiagonal matrices unitary equivalent with normal matrices were studied.
- A generalization of well-known methods for specific normal matrices.
  - Specific reduction types;
  - Krylov relations.
- Alternative computation of the SVD of normal matrices, starting point for further research.

# Important references

- L. Elsner;
- Kh. D. Ikramov;
- H. Fassbender;
- R. Grone & Johnsson & E. M. Sa;
- R. Horn & C. Johnsson;
- C. Mehl.