$\boxed{\mathcal{H}_2\text{-optimal approximation of linear dynamical systems}}$

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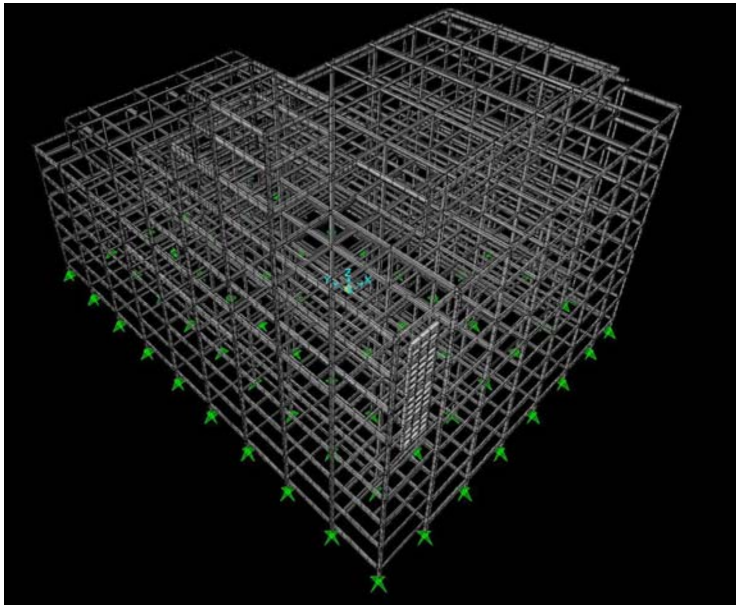
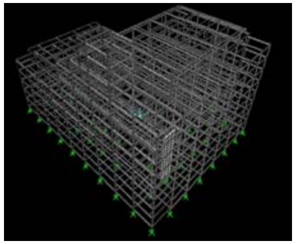
Model reduction simplifies complex models for simulation, optimization and control



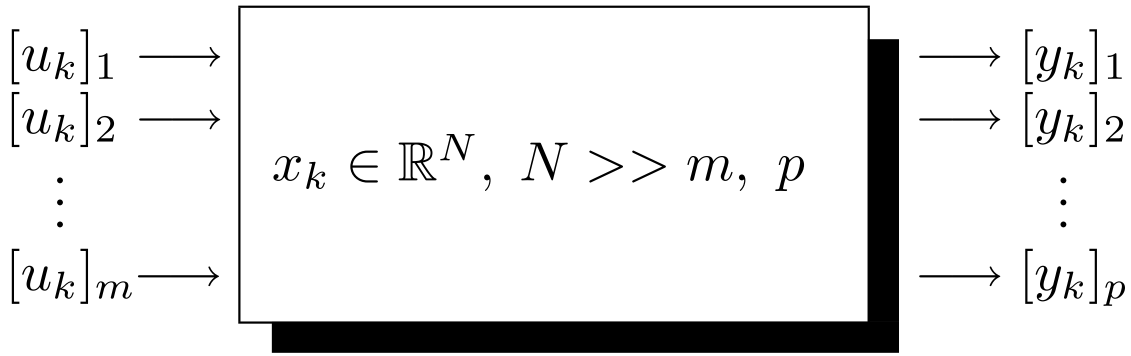


from 52800 to 40 differential equations

Dongting Lake Bridge has now MR dampers

to control (dampen) wind-induced vibration

We mainly look at discrete-time state-space models



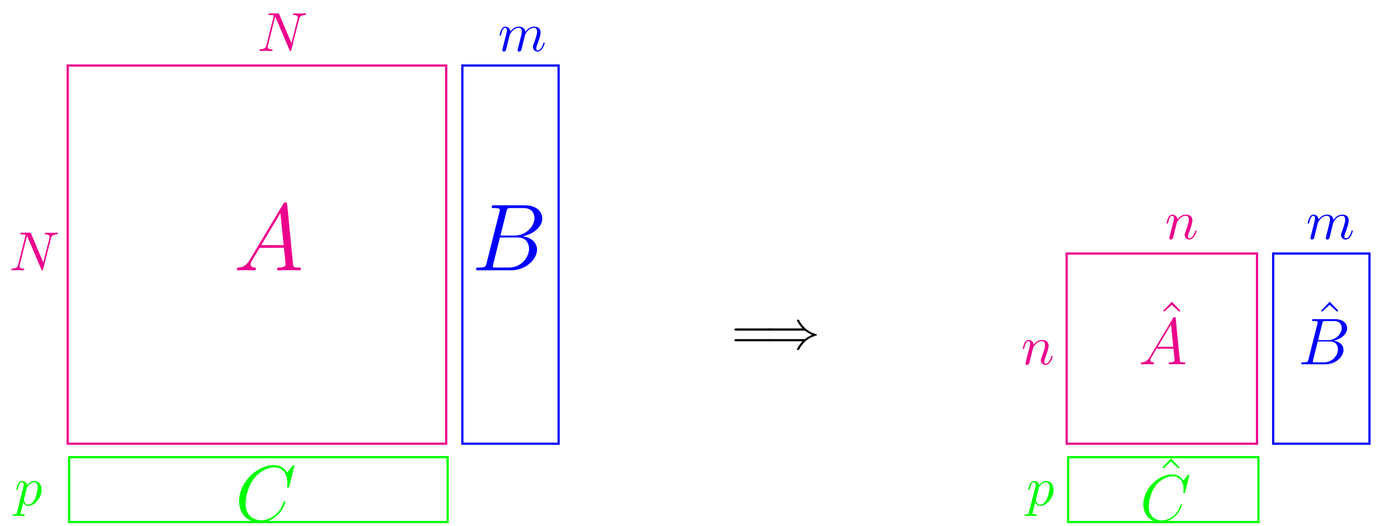
(explicit) time-varying

$$\begin{cases} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{cases}$$

(explicit) time-invariant

$$\begin{cases} x_{k+1} &= A x_k + B u_k \\ y_k &= C x_k \end{cases}$$

Time invariant model reduction idea



where $\quad n \ll N, \quad \hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV$

$$P = VW^T \quad \text{is a projector for } W^T V = I_n$$

$$E(z) := H(z) - \hat{H}(z) = C(zI_N - A)^{-1}B - \hat{C}(zI_n - \hat{A})^{-1}\hat{B}$$

Error model

The difference of the systems

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \\ y_k = Cx_k \end{cases} \quad \text{and} \quad \begin{cases} \hat{x}_{k+1} = \hat{A}\hat{x}_k + \hat{B}u_k \\ \hat{y}_k = \hat{C}\hat{x}_k \end{cases}$$

is the error model, where $e_k := y_k - \hat{y}_k$

$$\begin{cases} \tilde{x}_{k+1} = A_e\tilde{x}_k + B_e u_k \\ e_k = C_e\tilde{x}_k \end{cases}$$

with

$$(A_e, B_e, C_e) := \left(\begin{bmatrix} A & \\ & \hat{A} \end{bmatrix}, \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \begin{bmatrix} C & -\hat{C} \end{bmatrix} \right),$$

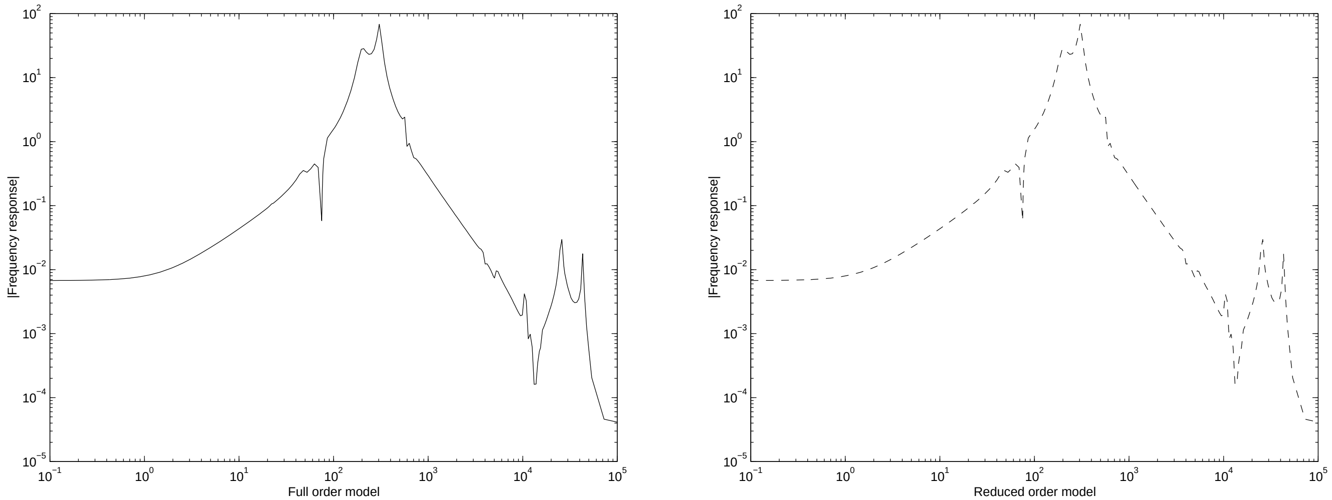
and transfer function

$$E(z) := H(z) - \hat{H}(z) = C_e(zI - A_e)^{-1}B_e$$

Frequency and time response matching

Minimizing the cost $\mathcal{J} := \|E(z)\|_{\mathcal{H}_2} := tr \int_{-\infty}^{\infty} E(e^{j\omega})E(e^{j\omega})^H \frac{d\omega}{2\pi}$

ensures the frequency response to match



and the time responses to match if $H(z)$ and $\hat{H}(z)$ are stable since

$$\mathcal{J} = \mathrm{tr} \sum_{k=0}^{\infty} (C_e A_e^k B_e)(C_e A_e^k B_e)^T$$

$\boxed{\text{How to evaluate this norm ?}}$

$$\mathcal{J} = \operatorname{tr}\left(C_e P_e C_e^T\right) = \operatorname{tr}\left(B_e^T Q_e B_e\right)$$

where P_e and Q_e solve the Stein equations

$$A_e P_e A_e^T + B_e B_e^T = P_e, \quad A_e^T Q_e A_e + C_e^T C_e = Q_e$$

One can also partition

$$P_e := \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix}, \quad Q_e := \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix}$$

and solve

$$\begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix} \begin{bmatrix} A^T & \\ & \hat{A}^T \end{bmatrix} + \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \begin{bmatrix} B^T & \hat{B}^T \end{bmatrix} = \begin{bmatrix} P & X \\ X^T & \hat{P} \end{bmatrix},$$

$$\begin{bmatrix} A^T & \\ & \hat{A}^T \end{bmatrix} \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix} \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} + \begin{bmatrix} C^T \\ -\hat{C}^T \end{bmatrix} \begin{bmatrix} C & -\hat{C} \end{bmatrix} = \begin{bmatrix} Q & Y \\ Y^T & \hat{Q} \end{bmatrix}$$

$\boxed{\text{Derive to maximize}}$

Let us define the gradient of a scalar function $f(X)$ as

$$[\nabla_X f(X)]_{i,j} = \frac{d}{dX_{i,j}} f(X), \quad i = 1, \ldots, n, \quad j = 1, \ldots, p$$

then the gradients $\nabla_{\hat{A}}\mathcal{J}$, $\nabla_{\hat{B}}\mathcal{J}, \nabla_{\hat{C}}\mathcal{J}$ satisfy the equations

$$\boxed{\tfrac{1}{2}\nabla_{\hat{A}}\mathcal{J} = \hat{Q}\hat{A}\hat{P} + Y^T A X, \quad \tfrac{1}{2}\nabla_{\hat{B}}\mathcal{J} = \hat{Q}\hat{B} + Y^T B, \quad \tfrac{1}{2}\nabla_{\hat{C}}\mathcal{J} = \hat{C}\hat{P} - C X}$$

where

$$A^T Y \hat{A} - C^T \hat{C} = Y, \quad \hat{A}^T \hat{Q} \hat{A} + \hat{C}^T \hat{C} = \hat{Q},$$

$$\hat{A} X^T A^T + \hat{B} B^T = X^T, \quad \hat{A}\hat{P}\hat{A}^T + \hat{B}\hat{B}^T = \hat{P}$$

Imposing zero gradients yields non-minimal optimality conditions!

Wilson 70, but rederived several times

Algorithm for minimizing $\|E(z)\|_{\mathcal{H}_2}$

Define $(X, Y, \hat{P}, \hat{Q}) = F(\hat{A}, \hat{B}, \hat{C})$ where

$$A^T Y \hat{A} - C^T \hat{C} = Y, \quad \hat{A}^T \hat{Q} \hat{A} + \hat{C}^T \hat{C} = \hat{Q},$$

$$\hat{A} X^T A^T + \hat{B} B^T = X^T, \quad \hat{A} \hat{P} \hat{A}^T + \hat{B} \hat{B}^T = \hat{P}$$

and then compute$(\hat{A}, \hat{B}, \hat{C}) = G(X, Y, \hat{P}, \hat{Q})$ from

$$W := -Y \hat{Q}^{-1}, \, V := X \hat{P}^{-1} \; \hat{A} = W^T A V, \, \hat{B} = W^T B, \, \hat{C} = C V,$$

The fixed point of $(\hat{A}, \hat{B}, \hat{C}) = G(F(\hat{A}, \hat{B}, \hat{C}))$ are also stationary points of $\|E(z)\|_{\mathcal{H}_2}$ and satisfy the interpolation conditions

One can also define a CG-like method or even a Newton-like method (see Antoulas-Sorenson, Beattie-Gugercin)

$\boxed{\text{Critical point conditions?}}$

But for first order poles

$$\hat{H}(z) = \sum_{i=1}^{n} \frac{\hat{c}_i \hat{b}_i^H}{z - \hat{\lambda}_i},$$

one obtains the interpolation conditions (where $H_*(z) := z^{-1} H^T(z^{-1})$)

$$[H_*(\hat{\lambda}_i) - \hat{H}_*(\hat{\lambda}_i)]\hat{c}_i = 0 \quad \hat{b}_i^H [H_*(\hat{\lambda}_i) - \hat{H}_*(\hat{\lambda}_i)] = 0$$

$$\hat{b}_i^H \frac{d}{dz} \left[H_*(z) - \hat{H}_*(z)\right]\Big|_{z=\hat{\lambda}_i} \hat{c}_i$$

Antoulas, Gugercin et al, Van Dooren et al, Bunse Gerstner et al

Follows also from gradient expressions and tangential interpolation (Gallivan-Vandendorpe-VD)

The first and second order case

Assume a real first order reduced model $\hat{H}(z) = \frac{cb^T}{z-\lambda}$
then the conditions become

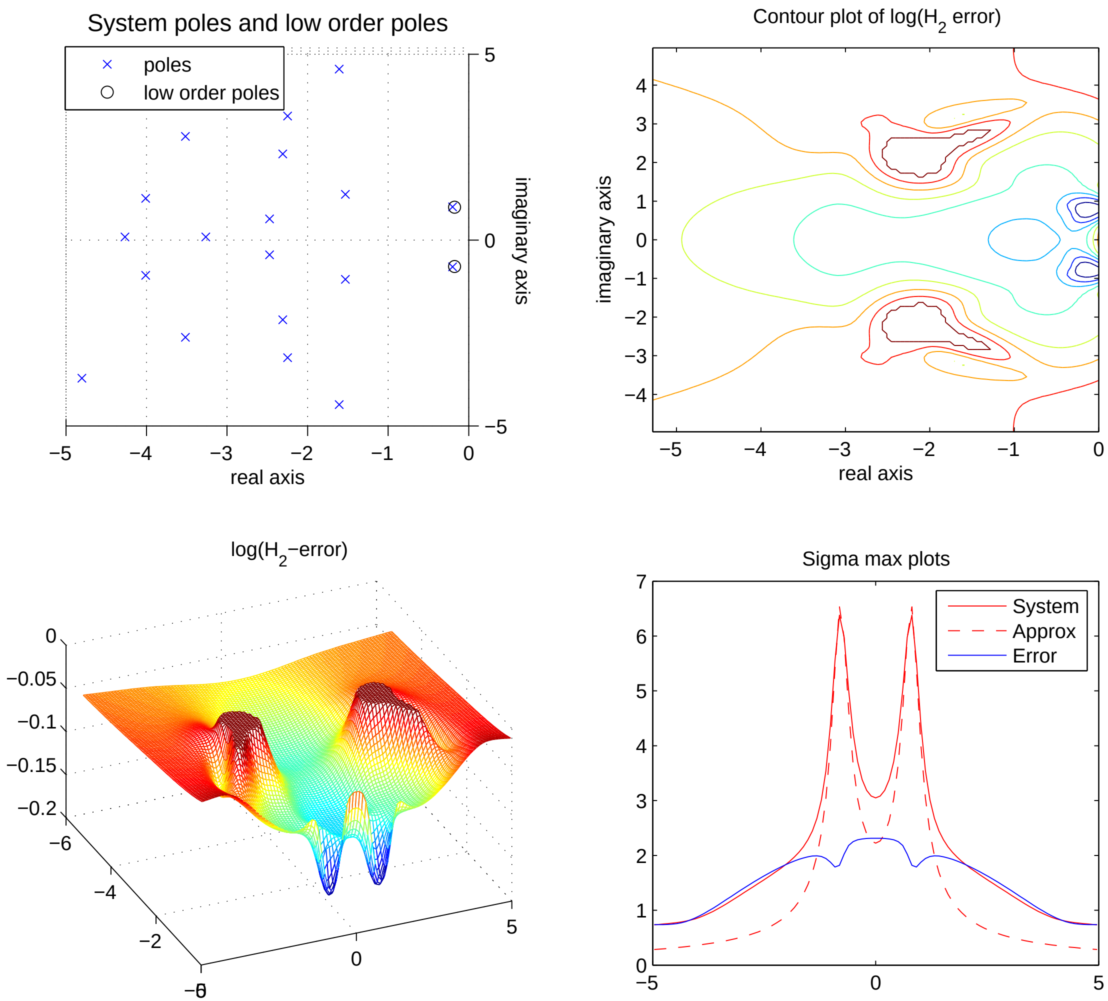
$$H_*(\lambda)c = b\frac{c^Tc}{\lambda^{-2}-1}, \quad b^T H_*(\lambda) = c^T\frac{b^Tb}{\lambda^{-2}-1},$$

This says that b and c must be the dominant singular vectors of $H_*(\lambda)$ and can be eliminated from the optimization problem

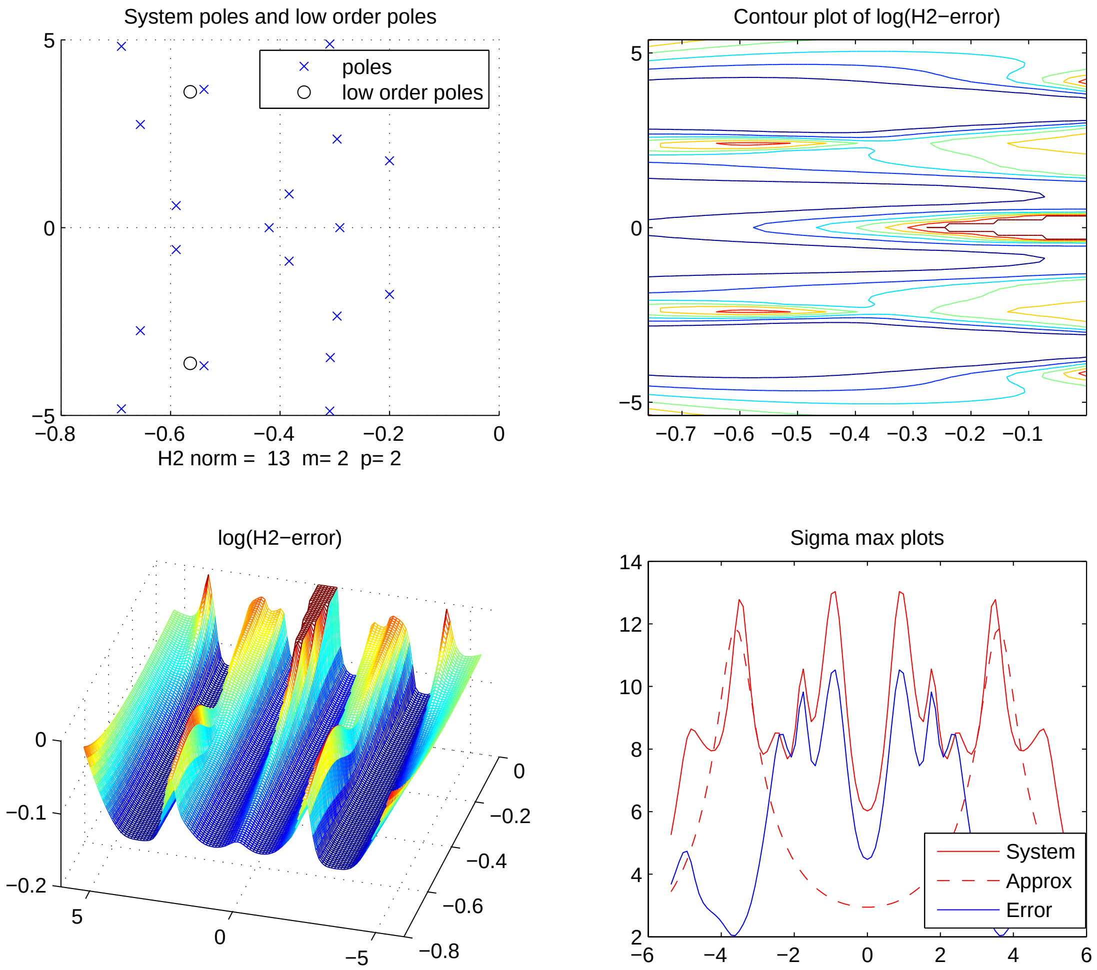
Assume a real second order reduced model $\hat{H}(z) = \frac{cb^H}{z-\lambda} + \frac{\overline{c}\overline{b}^H}{z-\overline{\lambda}}$
then the conditions again say that b and c must be the dominant singular vectors of $H_*(\lambda)$ and can be eliminated from the optimization problem

Now look at error $\|E(z)\|_{\mathcal{H}_2}$ as a function of interpolation point λ

MIMO example (CT, N=20, n=2, m=p=2)



MIMO example (CT, N=20, n=2, m=p=2)



$\boxed{\text{Conditions for higher order poles?}}$

For

$$\hat{H}(z) = \sum_{i=1}^{\ell} \hat{C}_i (zI - \hat{A}_i)^{-1} \hat{B}_i^H,$$

we obtain the following minimal interpolation conditions

$$[H_*(z) - \hat{H}_*(z)]\hat{c}_i(z) = O(z - \hat{\lambda}_i)^{k_i},$$

$$\hat{b}_i^H(z)[H_*(z) - \hat{H}_*(z)] = O(z - \hat{\lambda}_i)^{k_i},$$

$$\hat{b}_i^H(z)[H_*(z) - \hat{H}_*(z)]\hat{c}_i(z) = O(z - \hat{\lambda}_i)^{2k_i}$$

where $\quad \hat{b}_i^H(z) := \psi_{\hat{\lambda}_i}(z)\hat{B}_i^H, \quad \hat{c}_i(z) := \hat{C}_i \phi_{\hat{\lambda}_i}(z), \quad$ and

$$\phi_{\hat{\lambda}_i}(z) = \left[1, \ldots, (z - \hat{\lambda}_i)^{k-1}\right]^T, \quad \psi_{\hat{\lambda}_i}(z) = \left[(z - \hat{\lambda}_i)^{k-1}, \ldots, 1\right]$$

This does not follow from tangential interpolation conditions of Vandendorpe et al.

About continuity and sensitivity

On the negative side

Every stable and regular n-th degree $\hat{H}(z) = \hat{C}(zI_n - \hat{A})^{-1}\hat{B}$ is an optimal $\mathcal{H}_2$ approximation of some stable and regular N-th degree transfer function $H(z) = C(zI_N - A)^{-1}B$ for every $N > n$.

On the positive side

Let $\hat{H}(z)$ and $H(z)$ be stable and minimal and let $\hat{H}(z)$ be a stationary point of the $\mathcal{H}_2$ error function. Then every "nearby" transfer function $\hat{H}_\Delta(z)$ is a stationary point of a nearby system $H_\Delta(z)$. The same holds for every nondegenerate local minimum.

On the negative side again

Since minimal conditions are non-smooth around higher order poles, the interpolation problem becomes poorly conditioned in their neighborhood

$\boxed{\text{Time-varying case}}$

Systems now look like

$$\left\{\begin{array}{rl} x_{k+1} & = A_k x_k + B_k u_k \\ y_k & = C_k x_k \end{array}\right. \qquad \left\{\begin{array}{rl} \hat{x}_{k+1} & = \hat{A}_k \hat{x}_k + \hat{B}_k u_k \\ \hat{y}_k & = \hat{C}_k \hat{x}_k \end{array}\right.$$

with an error system where $e_k := y_k - \hat{y}_k$

$$\mathcal{E} := \left\{\begin{array}{rl} x^e_{k+1} & = A^e_k x^e_k + B^e_k u_k \\ e_k & = C^e_k x^e_k \end{array}\right.$$

where

$$A^e_k := \begin{bmatrix} A_k & \\ & \hat{A}_k \end{bmatrix}, \quad B^e_k = \begin{bmatrix} B_k \\ \hat{B}_k \end{bmatrix}, \quad C^e_k = \begin{bmatrix} C_k & -\hat{C}_k \end{bmatrix}$$

Its state for initial condition $x^e_{k_0} = 0$ is given by

$$x^e_k = \sum_{i=k_0}^{k-1} \Phi^e_{k,i+1} B^e_i u_i, \quad \Phi^e_{k+1,i} = A^e_k \Phi^e_{k,i} \; (k \geq i), \quad \Phi^e_{k,k} = I$$

Error system response satisfies

$$\tilde{e} = E\tilde{u}, \quad \tilde{e} := \begin{bmatrix} e_{k_0+1} \\ \vdots \\ e_{k_f+1} \end{bmatrix}, \quad \tilde{u} := \begin{bmatrix} u_{k_0} \\ \vdots \\ u_{k_f} \end{bmatrix}, \quad E = D_C H D_B$$

and

$$D_C = \begin{bmatrix} C^e_{k_0+1} & & 0 \\ & \ddots & \\ 0 & & C^e_{k_f+1} \end{bmatrix}, \quad D_B = \begin{bmatrix} B^e_{k_0} & & 0 \\ & \ddots & \\ 0 & & B^e_{k_f} \end{bmatrix}$$

$$H = \begin{bmatrix} \Phi^e_{k_0,k_0} & & 0 \\ \vdots & \ddots & \\ \Phi^e_{k_f,k_0} & \dots & \Phi^e_{k_f,k_f} \end{bmatrix}$$

The "stacked" error system response is $\tilde{e} = E\tilde{u}$ and the cost function to minimize is now given by

$$\|\mathcal{E}\|^2_{\mathcal{H}_2} := \mathcal{J}(k_0, k_f) := \mathrm{tr}(E^T E) = tr(EE^T)$$

One shows that

$$\mathcal{J}(k_0, k_f) := \mathrm{tr} \sum_{k=k_0+1}^{k_f+1} C^e_k P^e_k C^{e^T}_k = \mathrm{tr} \sum_{k=k_0}^{k_f} B^{e^T}_k Q^e_k B^e_k$$

where

$$P^e_{k+1} = \begin{bmatrix} A_k & \\ & \hat{A}_k \end{bmatrix} P^e_k \begin{bmatrix} A^T_k & \\ & \hat{A}^T_k \end{bmatrix} + \begin{bmatrix} B_k \\ \hat{B}_k \end{bmatrix} \begin{bmatrix} B^T_k & \hat{B}^T_k \end{bmatrix}, \quad P^e_{k_0} = 0$$

$$Q^e_{k-1} = \begin{bmatrix} A^T_k & \\ & \hat{A}^T_k \end{bmatrix} Q^e_k \begin{bmatrix} A_k & \\ & \hat{A}_k \end{bmatrix} + \begin{bmatrix} C^T_k \\ \hat{C}^T_k \end{bmatrix} \begin{bmatrix} C_k & \hat{C}_k \end{bmatrix}, \quad Q^e_{k_f+1} = 0$$

Gradients are now given by

$$\nabla_{\hat{A}_k}\mathcal{J} = 2(\hat{Q}_k\hat{A}_k\hat{P}_k + Y_k^T A_k X_k),$$

$$\nabla_{\hat{B}_k}\mathcal{J} = 2(\hat{Q}_k\hat{B}_k + Y_k^T B_k),$$

$$\nabla_{\hat{C}_k}\mathcal{J} = 2(\hat{C}_k\hat{P}_k - C_k X_k)$$

Updating rules and fixed point results are as before

$$W_k := Y_k\hat{Q}_k^{-1}, V_k = X_k\hat{P}_k^{-1}$$

$$(A_k^e, B_k^e, C_k^e) := (W_k^T A_k V_k, W_k^T B_k, C_k V_k).$$

where $X_k, \quad Y_k, \quad \tilde{P}_k, \quad \tilde{Q}_k$ satisfy Stein like recurrences

$$X_{k+1} = A_k X_k \hat{A}_k^T + B_k\hat{B}_k^T, \quad X_{k_0} = 0$$

$$\hat{P}_{k+1} = \hat{A}_k\hat{P}_k\hat{A}_k^T + \hat{B}_k\hat{B}_k^T, \quad \hat{P}_{k_0} = 0$$

$$Y_{k-1} = A_k^T Y_k \hat{A}_k^T - C_k^T\hat{C}_k, \quad Y_{k_f+1} = 0$$

$$\hat{Q}_{k-1} = \hat{A}_k^T Q_k \hat{A}_k + \hat{C}_k^T\hat{C}_k, \quad \hat{Q}_{k_f+1} = 0$$

Concluding remarks and references

- $\mathcal{H}_2$ model reduction allows for efficient optimization
  Gugercin-Beattie-Antoulas
- Interpolation of rational matrix functions
  Ball-Gohberg-Rodman, (OT45, Birkhauser 1990)
- Stationary points of time-invariant case amounts to interpolation
  Wilson, Gugercin-Beattie, Bunse Gerstner et al,
  VD-Gallivan-Absil
- Can be extended to discrete time-varying systems
  VD-Gallivan-Absil
- Time-varying systems have semi-separable Hankel maps
  Vanderveen-Dewilde
- Higher order case corresponds to Krylov methods
  Gallivan-Vandendorpe-VD, VD-Gallivan-Absil
- Sylvester equations are solved via rational Krylov
  Gallivan-Vandendorpe-VD