Spectral analysis and preconditioning in Finite Element approximations of elliptic PDEs

Cristina Tablino-Possio

joint work with Alessandro Russo

Dipartimento di Matematica e Applicazioni,
Università di Milano-Bicocca, Milano, Italy

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Outline

The Problem - Variational Form

Convection-Diffusion Equations

$$\begin{cases} \operatorname{div}\left(-a(\mathbf{x})\nabla u + \vec{\beta}(\mathbf{x})u\right) = f, \quad \mathbf{x} \in \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

Variational form

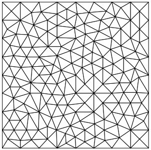
$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \left(a\nabla u \cdot \nabla\varphi - \vec{\beta}\cdot\nabla\varphi\ u\right) = \int_\Omega f\varphi \quad \text{for all } \varphi \in H_0^1(\Omega). \end{cases}$$

Regularity Assumptions

$$\begin{cases} a \in \mathbf{C}^2(\overline{\Omega}), & \text{with } a(\mathbf{x}) \geq a_0 > 0, \\ \vec{\beta} \in \mathbf{C}^1(\overline{\Omega}), & \text{with } \operatorname{div}(\vec{\beta}) \geq 0 \text{ pointwise in } \Omega, \\ f \in L^2(\Omega). \end{cases}$$

The Problem - FE Approximation - I

Let
$\mathcal{T}_h = \{K\}$ finite element partition of $\overline{\Omega}$, polygonal domain, into triangles,
$h_K = \mathrm{diam}(K)$,
$h = \max_K h_K$.



We consider the space of linear finite elements
$V_h = \{\varphi_h : \overline{\Omega} \to \mathbb{R}$ s.t. φ_h is continuous, $\varphi_{h|_K}$ is linear, and $\varphi_{h|\partial\Omega} = 0\} \subset H_0^1(\Omega)$

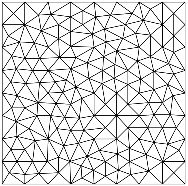
with basis

$$\varphi_i \in V_h \text{ s.t. } \varphi_i(\text{node } j) = \delta_{i,j},\ i,j = 1,\ldots,n(h),$$
$$n(h) = dim(V_h) = \text{ number of the internal nodes of } \mathcal{T}_h.$$

The Problem - FE Approximation - II

The variational equation becomes

$$A_n(a, \vec{\beta})\mathbf{u} = \mathbf{b}$$



$\mathcal{T}_h = \{K\}$

with

$$\begin{aligned} A_n(a, \vec{\beta}) = \sum_{K \in \mathcal{T}_h} A_n^K(a, \vec{\beta}) &= \Theta_n(a) + \Psi_n(\vec{\beta}) \in \mathbb{R}^{n \times n}, \quad n = n(h), \\ (\Theta_n(a))_{i,j} &= \sum_{K \in \mathcal{T}_h} \int_K a \, \nabla\varphi_j \cdot \nabla\varphi_i \qquad \text{diffusive term,} \\ (\Psi_n(\vec{\beta}))_{i,j} &= -\sum_{K \in \mathcal{T}_h} \int_K (\vec{\beta} \cdot \nabla\varphi_i) \, \varphi_j \qquad \text{convective term,} \end{aligned}$$

and with suitable quadrature formulas in the case of non constant a and $\vec{\beta}$.

Motivations

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- Recent attention to Hermitian/Skew-Hermitian splitting (HSS) iterations proposed in *Bai et al.* (2003) for non-Hermitian linear systems with positive definite real part: Bai, Benzi, Bertaccini, Gander, Golub, Ng, Serra-Capizzano, Simoncini, TP, ...
- Preconditioned HHS splitting iterations proposed in *Bertaccini et al.* (2005) for non-Hermitian linear systems with positive definite real part.
- Previously considered preconditioning strategy for FD/FE approximations of diffusion Eqns and FD approximations of Convection-Diffusion Eqns: Beckermann, Bertaccini, Golub, Serra-Capizzano, TP, ...

Aim

To study the effectiveness of the proposed Preconditioned HSS method applied to the FE approximations of Convection-Diffusion Eqns. both from the theoretical and numerical point of view.

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PHSS Method - Definition

Let us consider

$$A_n \mathbf{x} = \mathbf{b}, \quad A_n \in \mathbb{C}^{n\times n} \text{ with a positive definite real part}, \mathbf{x}, \mathbf{b} \in \mathbb{C}^n$$

The HSS method [1] refers to the Hermitian/Skew-Hermitian splitting

$$A_n = \mathrm{Re}(A_n) + \mathrm{i}\,\mathrm{Im}(A_n), \quad \mathrm{i}^2 = -1$$

with $\mathrm{Re}(A_n) = (A_n + A_n^H)/2$ and $\mathrm{Im}(A_n) = (A_n - A_n^H)/(2\mathrm{i})$ Hermitian matrices.

Here, we consider the Preconditioned HSS (PHSS) method [2]

$$\begin{cases} \left(\alpha I + P_n^{-1}\mathrm{Re}(A_n)\right)\mathbf{x}^{k+\frac{1}{2}} & = & \left(\alpha I - P_n^{-1}\mathrm{i}\,\mathrm{Im}(A_n)\right)\mathbf{x}^{k} + P_n^{-1}\mathbf{b} \\ \left(\alpha I + P_n^{-1}\mathrm{i}\,\mathrm{Im}(A_n)\right)\mathbf{x}^{k+1} & = & \left(\alpha I - P_n^{-1}\mathrm{Re}(A_n)\right)\mathbf{x}^{k+\frac{1}{2}} + P_n^{-1}\mathbf{b} \end{cases}$$

with P_n Hermitian positive definite matrix and α positive parameter.

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[1] Bai, Golub, Ng, SIMAX, 2003.

[2] Bertaccini, Golub, Serra-Capizzano, TP, Numer. Math., 2005.

PHSS Method - Convergence Properties

Theorem (Bertaccini *et al.*, 2005)

Let $A_n \in \mathbb{C}^{n\times n}$ be a matrix with positive definite real part, let α be a positive parameter and let $P_n \in \mathbb{C}^{n\times n}$ be a Hermitian positive definite matrix.
Then, the PHSS method is unconditionally convergent, since

$$\varrho(M(\alpha)) \leq \sigma(\alpha) = \max_{\lambda_i \in \lambda(P_n^{-1}\mathrm{Re}(A_n))} \left| \frac{\alpha - \lambda_i}{\alpha + \lambda_i} \right| < 1 \quad \textit{for any} \quad \alpha > 0,$$

with iteration matrix

$$M(\alpha) = \left(\alpha I + \mathrm{i}\, P_n^{-1}\mathrm{Im}(A_n)\right)^{-1} \left(\alpha I - P_n^{-1}\mathrm{Re}(A_n)\right) \left(\alpha I + P_n^{-1}\mathrm{Re}(A_n)\right)^{-1} \left(\alpha I - \mathrm{i}\, P_n^{-1}\mathrm{Im}(A_n)\right).$$

Moreover, the optimal α value that minimizes the quantity $\sigma(\alpha)$ equals

$$\alpha^* = \sqrt{\lambda_{\min}(P_n^{-1}\mathrm{Re}(A_n))\lambda_{\max}(P_n^{-1}\mathrm{Re}(A_n))} \textit{ and } \sigma(\alpha^*) = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$$

with $\kappa = \lambda_{\max}(P_n^{-1}\mathrm{Re}(A_n))/\lambda_{\min}(P_n^{-1}\mathrm{Re}(A_n))$ spectral condition number.

PHSS/IPHSS Method - Inexact Iterations

From a practical point of view, the PHSS method can also be interpreted as the original HSS method where the identity matrix is replaced by P_n, i.e.,

$$\begin{cases} (\alpha P_n + \mathrm{Re}(A_n))\, \mathbf{x}^{k+\frac{1}{2}} = (\alpha P_n - \mathrm{i}\ \mathrm{Im}(A_n))\, \mathbf{x}^{k} + \mathbf{b} \\ (\alpha P_n + \mathrm{i}\ \mathrm{Im}(A_n))\, \mathbf{x}^{k+1} = (\alpha P_n - \mathrm{Re}(A_n))\, \mathbf{x}^{k+\frac{1}{2}} + \mathbf{b}. \end{cases}$$

In practice, the two half-steps of the outer iteration can be computed by applying a PCG and a Preconditioned GMRES method, with preconditioner P_n.

The accuracy for the stopping criterion of these additional inner iterative procedures is chosen by taking into account the accuracy obtained by the current step of the outer iteration.

We denote by IPHSS method the described inexact PHSS iterations.

The Preconditioning Strategy - Definition - I

In the case of the considered FE approximation of Convection-Diffusion Eqns, the Hermitian/skew-Hermitian splitting is given by

$$\mathrm{Re}(A_n(a,\vec{\beta})) = \sum_{K\in\mathcal{T}_h} \mathrm{Re}(A_n^K(a,\vec{\beta})) = \Theta_n(a) + \mathrm{Re}(\Psi_n(\vec{\beta})) \ \text{ spd},$$

$$\mathrm{i}\,\mathrm{Im}(A_n(a,\vec{\beta})) = \mathrm{i} \sum_{K\in\mathcal{T}_h} \mathrm{Im}(A_n^K(a,\vec{\beta})) = \mathrm{i}\,\mathrm{Im}(\Psi_n(\vec{\beta})),$$

and can be performed on any single elementary matrix related to $\mathcal{T}_h$.
Notice that $\mathrm{Re}(\Psi_n(\vec{\beta})) = 0$ if $\mathrm{div}(\vec{\beta}) = 0$.

Lemma

Let

$$\{E_n(\vec{\beta})\}, \quad E_n(\vec{\beta}) := \mathrm{Re}(\Psi_n(\vec{\beta})).$$

Under the regularity assumptions, then it holds

$$\|E_n(\vec{\beta})\|_2 \le \|E_n(\vec{\beta})\|_\infty \le Ch^2,$$

with C absolute constant only depending on $\vec{\beta}(\mathbf{x})$ and Ω.

The Preconditioning Strategy - Definition - II

The considered preconditioning matrix sequence, proposed in [1], is defined as

$$\{P_n(a)\}, \quad P_n(a) = D_n^{\frac{1}{2}}(a) A_n(1,0) D_n^{\frac{1}{2}}(a)$$

where $D_n(a) = \mathrm{diag}(A_n(a,0))\mathrm{diag}^{-1}(A_n(1,0))$, i.e., the suitable scaled main diagonal of $A_n(a,0)$ and $A_n(a,0)$ equals $\Theta_n(a)$.

Notice that the preconditioner is tuned only with respect to the diffusion matrix $\Theta_n(a)$ owing to the PHSS convergence properties.

[1] Serra-Capizzano, Numer. Math., 1999.

Spectral Analysis - Structured Meshes - I

Let $\{A_n(a,\vec{\beta})\}$, $n = n(h)$ be the matrix sequence associated to a family $\{\mathcal{T}_h\}$, with decreasing parameter h.

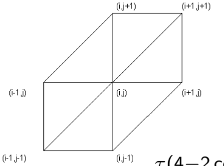
Aim:

- to quantify the difficulty of the linear system resolution vs the accuracy of the approximation scheme;
- to prove the optimality of the PHSS method.

We analyze the spectral properties of the preconditioned matrix sequences

$\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$ wrt PHSS/PCG

$\{P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))\}$ wrt PGMRES



$\tau(4-2\cos(s)-2\cos(t))$

in the special case of $\Omega = (0,1)^2$ with a structured uniform mesh.

Spectral Analysis - Structured Meshes - II

Theorem

Let $\{\mathrm{Re}(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the Hermitian positive definite matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$ is properly clustered at 1.
Moreover, for any n all the eigenvalues of $P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))$ belong to an interval $[d, D]$ well separated from zero (Spectral equivalence property).

The previous results prove the optimality both of the PHSS method and of the PCG, when applied in the IPHSS method for the inner iterations.

The proof technique refers to a previously analyzed FD case [1] and it is extended for dealing with the contribution given by $E_n(\vec{\beta})$.

[1] Serra-Capizzano, TP, ETNA, 2000; SIMAX 2003.

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Spectral Analysis - Structured Meshes - III

Theorem

Let $\{\mathrm{Im}(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the Hermitian matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))\}$ is spectrally bounded and properly clustered at 0 with respect to the eigenvalues.

The previous results prove that PGMRES converges superlinearly when applied to the matrix $I + \mathrm{i}P_n^{-1}(a)\mathrm{Im}(A_n)$ in the IPHSS method for the inner iterations.

The proof technique refers to the spectral Toeplitz theory and to the standard FE assembling procedure, according to a more natural local domain analysis approach [1].

[1] Beckermann, Serra-Capizzano, SINUM, 2007.

Spectral Analysis - Structured Meshes - III

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Spectral Analysis - Remarks

Remark

The previous Lemma and two Theorems hold both in the case in which the matrix elements are evaluated exactly and whenever a quadrature formula with error $O(h^2)$ is considered to approximate the involved integrals.

Complexity Issues - I

Definition (Axelsson, Neytcheva, 1994)

Let $\{A_m\mathbf{x}_m = \mathbf{b}_m\}$ be a given sequence of linear systems of increasing dimensions. An iterative method is *optimal* if

- the arithmetic cost of each iteration is at most proportional to the complexity of a matrix-vector product with matrix A_m,
- the number of iterations for reaching the solution within a fixed accuracy can be bounded from above by a constant independent of m.

Since

$$P_n(a) = D_n^{1/2}(a)A_n(1,0)D_n^{1/2}(a),$$

the solution of FE linear system with matrix $A_n(a, \vec{\beta})$ is reduced to computations involving diagonals and the matrix $A_n(1,0)$.

Complexity Issues - II

The latter task can be efficiently performed by means of fast Poisson solvers (e.g. cyclic reduction idea [1]) or several specialized algebraic multigrid methods [2] or geometric multigrid methods [3].

Therefore, for structured uniform meshes and under the regularity assumptions, the optimality of the PHSS method is theoretically proved.

The PHSS/IPHSS numerical performances do not worsen in the case of unstructured meshes.

[1] Buzbee, Dorr, George, Golub, SINUM, 1971.
[2] Serra-Capizzano, Numer. Math., 2002.
[3] Trottenberg, Oosterlee, Schüller, Academic Press, 2001.

Numerical Tests - PHSS/IPHSS Stopping Criterion

All the reported numerical experiments are performed in Matlab.
The outer iterative solvers starts with zero initial guess and the stopping criterion $||r_k||_2 \leq 10^{-7}||r_0||_2$ is considered for the outer iterations.

The numerical tests compare the PHSS and the IPHSS convergence properties.

In fact, a significant reduction of the computational costs can be obtained if the inner iterations are switched to the $(k+1)$–th outer step if

$$\frac{||r_{j,PCG}||_2}{||r_k||_2} \leq 0.1\,\eta^k, \quad \frac{||r_{j,PGMRES}||_2}{||r_k||_2} \leq 0.1\,\eta^k,$$

respectively, where k is the current outer iteration, $\eta \in (0,1)$, and where r_j is the residual at the j–th step of the present inner iteration.

Numerical Tests - Parameter α

The PHSS method is unconditionally convergent for any $\alpha > 0$. However, a suitable tuning can significantly reduce the number of outer iterations.

Clearly, the choice $\alpha = 1$ is evident whenever a cluster at 1 of the matrix sequence $\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$ is expected.

In the other cases, the target is to approximatively estimate the optimal α value

$$\alpha^* = \sqrt{\lambda_{\min}(P_n^{-1}\mathrm{Re}(A_n))\lambda_{\max}(P_n^{-1}\mathrm{Re}(A_n))}$$

that makes the spectral radius of the PHSS iteration matrix bounded by

$$\sigma(\alpha^*) = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1},$$

with $\kappa = \lambda_{\max}(P_n^{-1}\mathrm{Re}(A_n))/\lambda_{\min}(P_n^{-1}\mathrm{Re}(A_n))$ spectral condition number.

Numerical Tests - Structured Meshes - I

Test I: $a(x, y) = \exp(x + y)$, $\vec{\beta}(x, y) = [x\ y]^T$

n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
81	5	1.6 (8)	2.4 (12)	5	1 (5)	1 (5)
361	5	1.6 (8)	2.8 (14)	5	1 (5)	1 (5)
1521	5	1.6 (8)	3 (15)	5	1 (5)	2 (10)
6241	5	1.6 (8)	3.2 (16)	5	1 (5)	2 (10)
25281	5	1.6 (8)	3.6 (18)	5	1 (5)	2 (10)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

n	$P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))$					$P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))$				
	m_-	m_+	p_{tot}	$\lambda_{\min}$	$\lambda_{\max}$	m_-	m_+	p_{tot}	$\lambda_{\min}$	$\lambda_{\max}$
81	0	0	0%	9.99e-01	1.04e+00	0	0	0%	-2.68e-02	2.68e-02
361	0	0	0%	9.99e-01	1.04e+00	0	0	0%	-2.87e-02	2.87e-02
1521	0	0	0%	9.99e-01	1.044e+0	0	0	0%	-2.93e-02	2.93e-02

Outliers analysis ($\delta = 0.1$)

Numerical Tests - Structured Meshes - II

Test II: $a(x,y) = \exp(x + |y - 1/2|^{3/2})$, $\vec{\beta}(x,y) = [x \ y]^T$

n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
81	6	2.2 (13)	2.8 (17)	6	1 (6)	1 (6)
361	6	2.2 (13)	3.2 (19)	6	1 (6)	2 (12)
1521	6	2.2 (13)	3.5 (21)	6	1 (6)	2 (12)
6241	6	2.2 (13)	4 (24)	6	1 (6)	2 (12)
25281	6	2.2 (13)	4.2 (25)	6	1 (6)	3 (18)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

n	$P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))$					$P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))$				
	m_-	m_+	p_{tot}	$\lambda_{\min}$	$\lambda_{\max}$	m_-	m_+	p_{tot}	$\lambda_{\min}$	$\lambda_{\max}$
81	0	1	1.2%	9.97e-01	1.12e+00	0	0	0%	-4.32e-02	4.32e-02
361	0	1	0.27%	9.99e-01	1.12e+00	0	0	0%	-4.68e-02	-4.68e-02
1521	0	1	6%	9.99e-01	1.12e+00	0	0	0%	-4.78e-02	4.78e-02

Outliers analysis ($\delta = 0.1$)

Numerical Tests - Structured Meshes - III

Test III: $a(x,y) = \exp(x + |y - 1/2|)$, $\vec{\beta}(x,y) = [x\ y]^T$

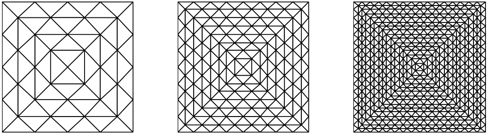
n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
81	7	1.9 (13)	2.6 (18)	7	1 (7)	1 (7)
361	7	2.2 (15)	3 (21)	7	1 (7)	1.7 (12)
1521	7	2.2 (15)	3.5 (24)	7	1.1 (8)	2 (14)
6241	7	2.3 (16)	3.6 (25)	7	1.1 (8)	2 (14)
25281	7	2.3 (16)	4 (28)	7	1.1 (8)	2.1 (15)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

n	$P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))$					$P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))$				
	m_-	m_+	p_{tot}	$\lambda_{\min}$	$\lambda_{\max}$	m_-	m_+	p_{tot}	$\lambda_{\min}$	$\lambda_{\max}$
81	0	1	1.2%	9.95e-01	1.16e+000	0	0	0%	-3.97e-02	3.97e-02
361	0	1	0.28%	9.97e-01	1.17e+00	0	0	0%	-4.31e-02	4.31e-02
1521	0	1	0.07%	9.98e-01	1.18e+00	0	0	0%	-4.40e-02	4.40e-02

Outliers analysis ($\delta = 0.1$)

Numerical Tests - Other Structured Meshes - I



Other Structured Meshes

$a(x,y) = \exp(x+y),\ \vec{\beta}(x,y) = [x\ y]^T$						
n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
25	5	2.2 (11)	2.2(11)	5	1 (5)	1 (5)
113	5	2.2 (11)	2.4 (12)	5	1 (5)	1 (5)
481	5	2.2 (11)	2.8 (14)	5	1 (5)	1 (5)
1985	5	2.2 (11)	3 (15)	5	1 (5)	2 (10)
8065	5	2.2 (11)	3.4 (17)	5	1 (5)	2 (10)
32513	5	2.2 (11)	3.6 (18)	5	1 (5)	2 (10)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

Numerical Tests - Other Structured Meshes - II

$a(x,y) = \exp(x + \|y - 1/2\|^{3/2})$, $\vec{\beta}(x,y) = [x\ y]^T$						
n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
25	6	2.2 (13)	2.5 (15)	6	1 (6)	1 (6)
113	6	2.2 (13)	3 (18)	6	1 (6)	1 (6)
481	6	2.2 (13)	3.2 (19)	6	1 (6)	2 (12)
1985	6	2.2 (13)	3.7 (22)	6	1 (6)	2 (12)
8065	6	2.2 (13)	4 (24)	6	1 (6)	2 (12)
32513	6	2.2 (13)	4.3 (26)	6	1 (6)	3 (18)

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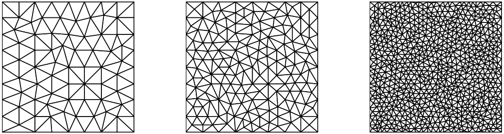
25	6	2.2 (13)	2.5 (15)	6	1 (6)	1 (6)
113	7	2 (14)	2.7 (19)	7	1 (7)	1 (7)
481	7	2.1 (15)	3 (21)	7	1.1 (8)	1.8 (13)
1985	7	2.3 (16)	3.4 (24)	7	1.1 (8)	2 (14)
8065	7	2.4 (17)	3.8 (27)	7	1.1 (8)	2 (14)
32513	8	2.1 (17)	3.8 (30)	8	1.1 (9)	2.1 (17)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

Numerical Tests - Unstructured Meshes - I



Unstructured Meshes

$a(x,y) = \exp(x+y),\ \vec{\beta}(x,y) = [x\ y]^T$						
n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
55	5	2.2 (11)	2.2 (11)	5	1 (5)	1 (5)
142	5	2.2 (11)	2.6 (13)	5	1 (5)	1 (5)
725	5	2.2 (11)	3 (15)	5	1 (5)	1 (5)
1538	5	2.2 (11)	3 (15)	5	1.2 (6)	1.2 (6)
7510	5	2.2 (11)	3 (17)	5	1.2 (6)	1.8 (9)
15690	5	2.2 (12)	3.4 (18)	6	1.2 (7)	2 (12)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

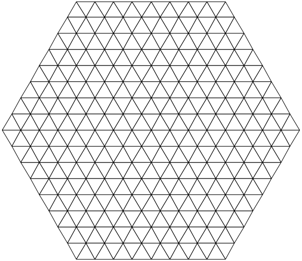
Numerical Tests - Unstructured Meshes - II

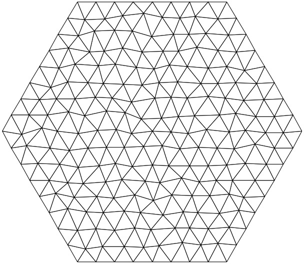
$a(x,y) = \exp(x + \|y - 1/2\|^{3/2}),\ \vec{\beta}(x,y) = [x\ y]^T$						
n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
55	6	2.2 (13)	2.7 (16)	6	1 (6)	1 (6)
142	6	2.2 (13)	3 (18)	6	1 (6)	1 (6)
725	6	2.2 (13)	3.3 (20)	6	1 (6)	2 (12)
1538	6	2.2 (13)	3.5 (21)	6	1.2 (7)	2 (12)
7510	7	2 (14)	3.7 (26)	7	1.1 (8)	2 (14)
15690	7	2 (14)	3.7 (26)	7	1.1 (8)	2 (14)

$a(x,y) = \exp(x + \|y - 1/2\|),\ \vec{\beta}(x,y) = [x\ y]^T$						
n	PHSS	PCG	PGMRES	IPHSS	PCG	PGMRES
55	7	1.8 (13)	2.4 (17)	7	1 (7)	1 (7)
142	7	2.2 (15)	2.8 (20)	7	1 (7)	1 (7)
725	7	2.6 (18)	3.1 (22)	7	1.1 (8)	2 (14)
1538	7	2.6 (18)	3.4 (24)	7	1.1 (8)	2 (14)
7510	7	2.6 (18)	3.7 (26)	7	1.1 (8)	2 (14)
15690	8	2.4 (19)	3.6 (29)	8	1.1 (9)	2 (16)

Number of PHSS/IPHSS outer iterations and average per outer step for PCG and PGMRES inner iterations (total number of inner iterations in brackets)

Some Perspectives





$a(x,y) = \exp(x+y)$		
n	PCG - $P_n(a)$	PCG - $\tilde{P}_n(a)$
37	4	9
169	4	10
721	4	11
2977	4	12
12097	4	12

Number of PCG iterations for Diffusion Eqns.

$$P_n(a) = D_n^{\frac{1}{2}}(a) A_n(1,0) D_n^{\frac{1}{2}}(a)$$
$$\tilde{P}_n(a) = D_n^{\frac{1}{2}}(a) \tilde{T}_n D_n^{\frac{1}{2}}(a),$$
$$\tilde{T}_n = \Pi T_m \Pi,$$
$$T_m = T_m(6 - 2\cos(s) - 2\cos(t) - 2\cos(s+t))$$

Conclusions

- The PHSS and IPHSS methods have the same convergence features, but the cost per iteration of the latter is substantially reduced.

- With the choice $\alpha = 1$ and under the regularity assumptions the PHSS method is optimally convergent, i.e., with a convergence rate independent of the matrix dimension $n(h)$.

- In the inner IPHSS iteration
  - the PCG converges superlinearly owing to the proper cluster at 1 of the matrix sequence $\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$, that induces a proper cluster at 1 for the sequence $\{I + P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$;

  - the PGMRES converges superlinearly when applied to the coefficient matrices $\{I + \mathrm{i}P_n^{-1}(a)\mathrm{Im}(A_n)\}$, owing to the spectral boundeness and the proper clustering at 0 of the sequence $\{P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))\}$.