Tools for analyzing the Spectral Distribution in a non Hermitian context

Debora Sesana

Department of Physics and Mathematics

University of “Insubria” - Como

Contents

Contents

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Contents

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Definition. The matrix sequence $\{A_n\}$ is *distributed in the sense of the eigenvalues* as the function θ on the set K (in symbols $\{A_n\} \sim_\lambda (\theta, K)$) if

$$\lim_{n\to\infty} \Sigma_\lambda(F, A_n) = \frac{1}{\mu\{K\}} \int_K F\left(\theta(s)\right)\, \mathrm{d}s, \quad \forall F \in \mathcal{C}_c(\mathbb{C}).$$

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- applications in statistics (Bercu, Gamboa,...);
- support for wireless communications (Gutierrez, Crespo, Najim, Gray,...);

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$$a_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-\hat{\imath} j s} \mathrm{d}s, \qquad \hat{\imath}^2 = -1, \qquad j \in \mathbb{Z}.$$

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The *Toeplitz matrix* $T_n(f)$ is defined in this way

$$T_n(f) = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-(n-2)} & a_{-(n-1)} \\ a_1 & \ddots & \ddots & \ddots & a_{-(n-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{pmatrix}$$

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f is known as a symbol or generating function of $T_n(f)$.
If f is real valued function then the matrix $T_n(f)$ is Hermitian, i.e. $a_j = \overline{a_{-j}}$.

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Theorem. Let f be a real valued integrable function over $Q = [-\pi, \pi)$, if $\{T_n(f)\}$ is the sequence of Toeplitz matrices generated by f, then it holds

$$\{T_n(f)\} \sim_\lambda (f, Q).$$

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Tools for approximation:

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- definition of approximating class of sequences;

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Tools for approximation:

- definition of approximating class of sequences;
- main theorem of distribution;

Tools for approximation: $a.c.s.$

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Definition. Let $\{A_n\}$ a given sequence of matrices, $A_n \in M_{d_n}(\mathbb{C})$, $d_n < d_{n+1}$.
$\{\{B_{n,m}\}\}_m$, $m \in \mathbb{N}$ is an *approximating class of sequences* *($a.c.s.$)* for $\{A_n\}$ if

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad \forall n > n_m, \forall m \in \mathbb{N},$$

$$\mathrm{Rank}(R_{n,m}) \leq d_n c(m), \quad \|N_{n,m}\| \leq w(m),$$

where $n_m \geq 0, c(m)$ and $w(m)$ are functions that depend only on m and

$$\lim_{m\to\infty} w(m) = 0, \qquad \lim_{m\to\infty} c(m) = 0.$$

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$$\begin{array}{ccc} \{\{B_{n,m}\}\}_m & \xrightarrow[\text{assumption 1}]{a.c.s.} & \{A_n\} \\ \sim_\lambda \Big\downarrow \text{assumption 2} & & \\ \{f_m\}_m & & \end{array}$$

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then θ is real valued and $\{A_n\} \sim_\lambda (\theta, G)$.

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Definition. A matrix sequence $\{A_n\}$ ($A_n \in M_{d_n}(\mathbb{C})$, $d_n < d_{n+1}$) is *weakly clustered* at a non empty closed set $S \subset \mathbb{C}$ (in the eigenvalue sense) if for any $\epsilon > 0$

$$\#\{j : \lambda_j(A_n) \notin D(S,\epsilon)\} = o(d_n), \quad n \to \infty,$$

where $D(S,\epsilon) := \bigcup_{s \in S} D(s,\epsilon)$ and $D(s,\epsilon) := \{z : |z - s| < \epsilon\}$.

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