Efficient reconstruction of functions on the sphere from scattered data

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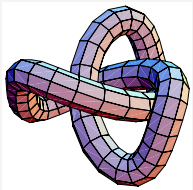


Content

- NFFT
- NFSFT
- Iterative reconstruction on $\mathbb{S}^2$
- Probabilistic arguments

numerical examples

NFFT (Dutt,Rokhlin;Beylkin; Steidl; P.,Steidl,Tasche)

NFFT

fast computation of of the sums

$$f(\boldsymbol{v}_j) = \sum_{k_1=-N/2}^{N/2-1} \cdots \sum_{k_d=-N/2}^{N/2-1} f_{\boldsymbol{k}} \, \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{v}_j} \qquad (j = -M/2, \ldots, M/2-1)$$

$$h(\boldsymbol{k}) = \sum_{j=-M/2}^{M/2-1} f_j \, \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k} \boldsymbol{v}_j} \qquad (-\frac{N}{2} \leq \boldsymbol{k} < \frac{N}{2})$$

for equispaced nodes $\boldsymbol{v}_j := \frac{\boldsymbol{j}}{N}$ $(M = N^d)$

FFT (*fast Fourier transform*) in $\mathcal{O}(N^d \log N)$

for arbitrary nodes $\boldsymbol{v}_j \in [-1/2, 1/2)^d$

NFFT (*nonequispaced FFT*) in $\mathcal{O}(N^d \log N + m^d M)$

Fourier algorithms on the sphere

Problem: fast computation of

$$f(\theta,\phi) = \sum_{k=0}^{N} \sum_{n=-k}^{k} a_k^n Y_k^n(\theta,\phi)$$

at arbitrary nodes $(\theta_d, \phi_d) \in \mathbb{S}^2$ $(d = 0, \ldots, M-1)$

- discrete spherical Fourier transform (FFT on $\mathbb{S}^2$)

$$(\theta_{d_1}, \phi_{d_2}) := \left(\tfrac{d_1\pi}{D_1}, \tfrac{2d_2\pi}{D_2-1}\right) \quad d_1 = 0, \ldots, D_1-1,\ d_2, \ldots, D_2-1$$

Driscoll, Healy (1994, 2003, ...); Potts, Steidl, Tasche (1998); Mohlenkamp (1999); Suda, Takami (2002); Rokhlin, Tygert (2006)

- nonequispaced discrete spherical Fourier transform (NFFT on $\mathbb{S}^2$)

$$(\theta_d, \phi_d) \in \mathbb{S}^2, \quad d = 0, \ldots, M-1$$

Kunis, P. (2003); Keiner, P. (2008)

Inverse NFFT on the sphere

$$f = \sum_{k=0}^{N} \sum_{n=-k}^{k} \hat{f}_k^n Y_k^n \quad \in \Pi_N(\mathbb{S}^2)$$

”inverse” problem, $\boldsymbol{f} \in \mathbb{C}^M$ given in

$$\begin{aligned} \boldsymbol{Y}\hat{\boldsymbol{f}} &\approx \boldsymbol{f}, \qquad \hat{\boldsymbol{f}} = \left(\hat{f}_k^n\right)_{k=0,\ldots,N,|n|\leq k} \in \mathbb{C}^{(N+1)^2},\ \boldsymbol{f} \in \mathbb{C}^M \\ \left(f(\boldsymbol{\xi}_j)\right)_{j=0,\ldots,M-1} &\approx \boldsymbol{f} \end{aligned}$$

spherical Fourier matrix

$$\boldsymbol{Y} := \left(Y_k^n\left(\boldsymbol{\xi}_j\right)\right)_{j=0,\ldots,M-1;k=0,\ldots,N,|n|\leq k} \in \mathbb{C}^{M\times(N+1)^2}.$$

The *geodetic distance* of $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^2$ is given by

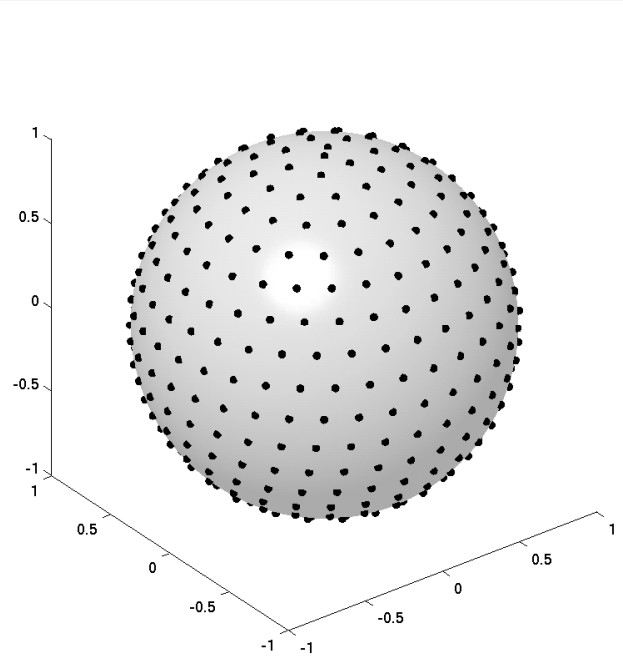
$$\text{dist}\,(\boldsymbol{\xi}, \boldsymbol{\eta}) := \arccos\,(\boldsymbol{\eta} \cdot \boldsymbol{\xi})\,.$$

We measure the "nonuniformity" of a *sampling set* $\mathcal{X} := \{\boldsymbol{\xi}_j \in \mathbb{S}^2 : j = 0, \ldots, M-1\}$, $M \in \mathbb{N}$, by the mesh norm $\delta_{\mathcal{X}}$ and the separation distance $q_{\mathcal{X}}$, defined by

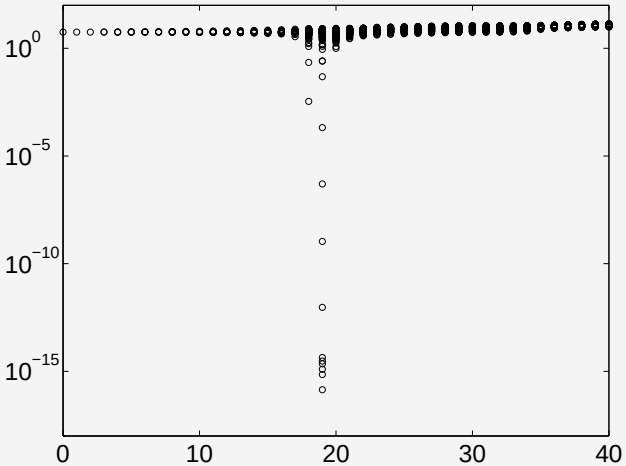
$$\delta_{\mathcal{X}} := 2 \max_{\boldsymbol{\xi} \in \mathbb{S}^2} \min_{j=0,\ldots,M-1} \text{dist}(\boldsymbol{\xi}_j, \boldsymbol{\xi}),$$
$$q_{\mathcal{X}} := \min_{0 \le j < l < M} \text{dist}(\boldsymbol{\xi}_j, \boldsymbol{\xi}_l).$$

The sampling set $\mathcal{X}$ is called

- δ-dense for some $0 < \delta \le 2\pi$, if $\delta_{\mathcal{X}} \le \delta$, and
- q-separated for some $0 < q \le 2\pi$, if $q_{\mathcal{X}} \ge q$.



Generalised spiral nodes



Distribution of the singular values of the spherical Fourier matrix $\boldsymbol{Y} \in \mathbb{C}^{M\times(N+1)^2}$ with respect to the polynomial degrees $N = 0, \ldots, 40$ for $M = 400$ generalised spiral nodes

Least squares approximation

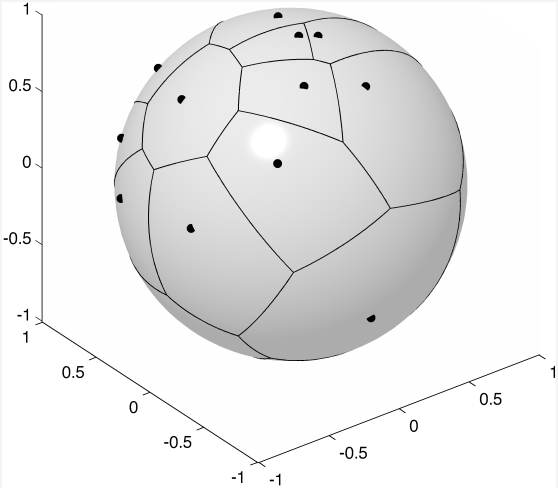
$M > (N+1)^2$ over-determined

$$\|\boldsymbol{f} - \boldsymbol{Y}\hat{\boldsymbol{f}}\|_{\boldsymbol{W}}^2 = \sum_{j=0}^{M-1} w_j |f_j - f(\boldsymbol{\xi}_j)|^2 \xrightarrow{\hat{\boldsymbol{f}}} \min$$

$\boldsymbol{W} :=$ diag$(w_j)_{j=0,\ldots,M-1} \in \mathbb{R}^{M\times M}$, weights $w_j > 0$

The least squares problem is equivalent to the normal equation of first kind

$$\boldsymbol{Y}^{\vdash}\boldsymbol{W}\boldsymbol{Y}\hat{\boldsymbol{f}} = \boldsymbol{Y}^{\vdash}\boldsymbol{W}\boldsymbol{f}.$$



Theorem: (Filbir ,Themistoclakis, 2008)
Let a δ-dense sampling set $\mathcal{X} \subset \mathbb{S}^2$ of cardinality $M \in \mathbb{N}$ be given. Moreover let for $N \in \mathbb{N}$ with $154N\delta < 1$ and $\boldsymbol{W} = \text{diag}(w_j)_{j=0,\ldots,M-1}$, with Voronoi weights w_j be given. Then we have for arbitrary spherical polynomials $f \in \Pi_N(\mathbb{S}^2)$, for the vector $\boldsymbol{f} = \left(f(\boldsymbol{\xi}_j)\right)_{j=0,\ldots,M-1}$ the weighted norm estimate

$$(1 - 154N\delta)\,\|f\|_{L^2}^2 \leq \|\boldsymbol{f}\|_{\boldsymbol{W}}^2 \leq (1 + 154N\delta)\,\|f\|_{L^2}^2\,.$$

Proof: based on spherical Marcinkiewicz-Zygmund inequalities (Mhaskar, Narcowich and Ward, 01; Filbir and Themistoclakis, 06).

Corollary:

$$1 - 154N\delta \leq \lambda_{\min}\left(\boldsymbol{Y}^{\vdash}\boldsymbol{W}\boldsymbol{Y}\right) \leq 1 \leq \lambda_{\max}\left(\boldsymbol{Y}^{\vdash}\boldsymbol{W}\boldsymbol{Y}\right) \leq 1 + 154N\delta$$

i.e.
a constant number of iterations in CGNR method is suffices to decrease the residual to a certain fraction

Optimal interpolation

$M < (N+1)^2$ under-determined

- given sample values $f_j \in \mathbb{C}$, $j = 0, \ldots, M-1$ and weights $\hat{w}_k > 0$

$$\min_{\hat{\boldsymbol{f}} \in \mathbb{C}^{(N+1)^2}} \sum_{k=0}^{N} \sum_{n=-k}^{k} \frac{\left|\hat{f}_k^n\right|^2}{\hat{w}_k} \quad \text{subject to} \quad \sum_{k=0}^{N} \sum_{n=-k}^{k} \hat{f}_k^n Y_k^n\left(\boldsymbol{\xi}_j\right) = f_j$$

The optimal interpolation problem is equivalent to the normal equations of second kind

$$\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\vdash}\tilde{\boldsymbol{f}} = \boldsymbol{f}, \qquad \hat{\boldsymbol{f}} = \hat{\boldsymbol{W}}\boldsymbol{Y}^{\vdash}\tilde{\boldsymbol{f}},$$

where $\hat{\boldsymbol{W}} := \text{diag}(\tilde{\boldsymbol{w}})$ with $\tilde{w}_k^n = \hat{w}_k$, $k = 0, \ldots, N$, $|n| \leq k$.

- polynomial kernel $K_N : [-1, 1] \to \mathbb{C}$ and its associated matrix

$$K_N(t) := \sum_{k=0}^{N} \frac{2k+1}{4\pi} \hat{w}_k P_k(t), \qquad \boldsymbol{K} := \left(K_N\left(\boldsymbol{\xi}_j \cdot \boldsymbol{\xi}_l\right)\right)_{j,l=0,\ldots,M-1}$$

$$\boldsymbol{K} = \boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\vdash}$$

Theorem: (Kunis, 2005; Keiner, Kunis, P., 2006)
Let a q-separated sampling set $\mathcal{X} \subset \mathbb{S}^2$ of cardinality $M \in \mathbb{N}$ and with $q \leq \pi$ be given. Then for $N, \beta \in \mathbb{N}$, $N \geq \beta - 1 \geq 2$, the kernel matrix

$$\boldsymbol{K} = (K_{j,l})_{j,l=0,\ldots,M-1}, \qquad K_{j,l} = B_{\beta,N}(\boldsymbol{\xi}_j \cdot \boldsymbol{\xi}_l),$$

has bounded eigenvalues

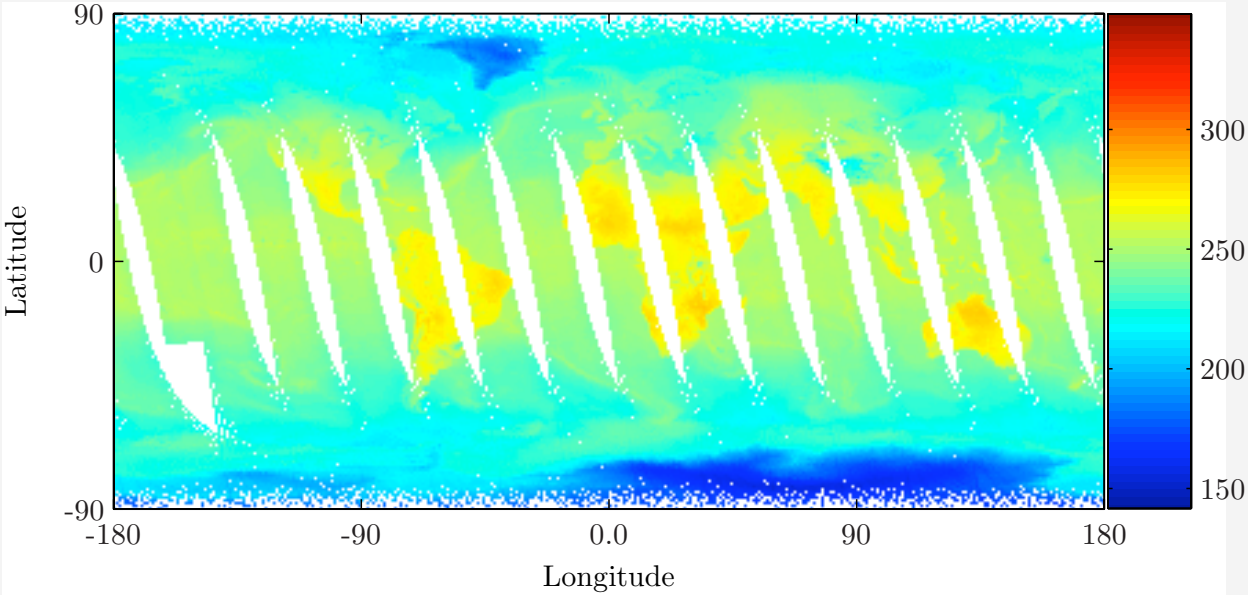
$$|\lambda(\boldsymbol{K}) - 1| \leq \frac{25 c_\beta \zeta\,(\beta - 1)}{((N+1)\,q)^\beta}.$$

Corollary: Let a q-separated sampling set $\mathcal{X} \subset \mathbb{S}^2$ of cardinality $M \in \mathbb{N}$ and with $q \leq \pi$ be given. Moreover, let $N \in \mathbb{N}$, $(N+1)q > 11.2$, and weights be given by the sampled cubic B-Spline. Then we have

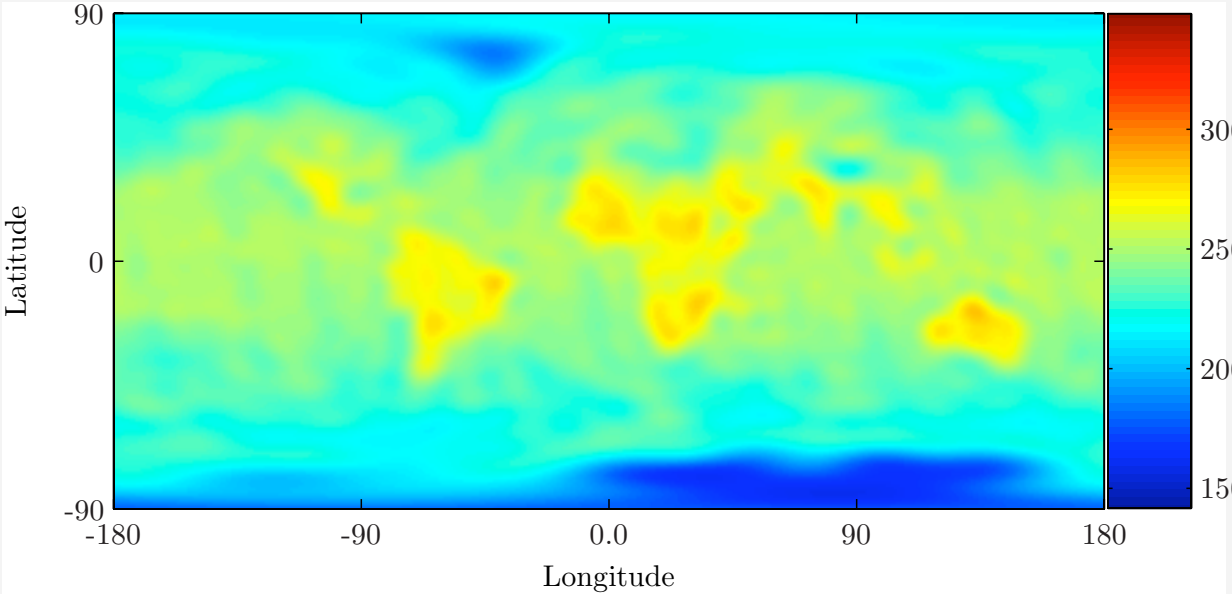
$$1 - \left(\frac{11.2}{(N+1)q}\right)^4 \leq \lambda_{\min}(\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\vdash}) \leq \lambda_{\max}(\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\vdash}) \leq 1 + \left(\frac{11.2}{(N+1)q}\right)^4.$$

i.e.
a constant number of iterations in CGNE method is suffices to decrease the error to a certain fraction

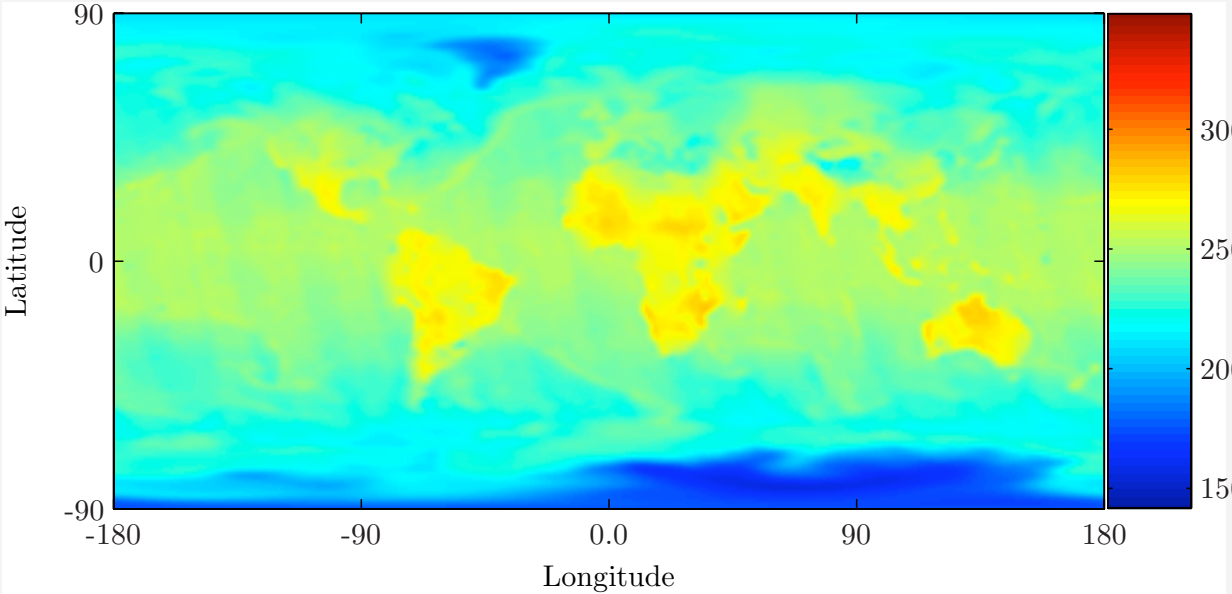
Numerical example



The original atmospheric temperature of the earth from 5 November 2006 measured by a satellite in Kelvin.



Least squares approximation to the global temperature data with $N = 32$.



Least squares approximation to the global temperature data with $N = 128$.

Probabilistic Marcinkiewicz-Zygmund Inequalities

(Böttcher, Kunis, P. 08)

Observation: in practice theoretically ill-conditioned systems often behave better than one would expect

Idea: probabilistic arguments

- Bass, Gröchenig, Rauhut (03,07): results for randomly chosen sampling nodes

- deterministic MZ inequality (Filbir ,Themistoclakis)

$$(1 - 154N\delta) \|f\|_{L^2}^2 \leq \|\boldsymbol{f}\|_{\boldsymbol{W}}^2 \leq (1 + 154N\delta) \|f\|_{L^2}^2$$

- Now: randomly chosen polynomials

$$\mathbb{P}\left((1 - \epsilon) \|f\|_{L^2}^2 \leq \|\boldsymbol{f}\|_{\boldsymbol{W}}^2 \leq (1 + \epsilon) \|f\|_{L^2}^2\right) \geq 1 - \eta$$

Example on $\mathbb{S}^2$:

- to ensure the inequality

$$\frac{1}{2}\|f\|_2 \leq \|\boldsymbol{f}\|_{\boldsymbol{W},2} \leq \frac{3}{2}\|f\|_2 \tag{1}$$

for $N \leq 13$, one has to require that $R \leq 1/(2 \cdot 13 \cdot 84) \approx 0.000\,46$ (Filbir, Themistoclakis 06, case q=3, p=2);

- $\hat{f}_k^n$ taken at random from the uniform distribution, then

$$\mathbb{P}\left(\frac{1}{2}\|f\|_2 \leq \|\boldsymbol{f}\|_{\boldsymbol{W},2} \leq \frac{3}{2}\|f\|_2\right) \geq 0.95$$

whenever $R \leq 0.000\,46$ and $N \leq 2\,184$, on the earth $l = 2.03km$, using (1) for $N \leq 2184$ we have to take $l = 12m$.

R is partition norm

$$R = \max_j \operatorname{diam} R_j := \max_j \max_{\boldsymbol{\xi},\boldsymbol{\eta} \in R_j} d(\boldsymbol{\xi}, \boldsymbol{\eta})$$

By A. Böttcher, S. Grudsky (EJP,03) it was shown that if $\boldsymbol{x}$ is randomly drawn from the uniform distribution and $\boldsymbol{A} \in \mathbb{C}^{m\times n}$

$$\mathbb{P}\left(\left|\frac{\|\boldsymbol{A}\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2} - \frac{\|\boldsymbol{A}\|_F^2}{n}\right| \leq \varepsilon\right) \geq 1 - \frac{2}{n\varepsilon^2}\left(\frac{\|\boldsymbol{A}\boldsymbol{A}^{\vdash}\|_F^2}{n} - \left(\frac{\|\boldsymbol{A}\|_F^2}{n}\right)^2\right)$$

Corollary: Let $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and suppose $\|\boldsymbol{A}\|_{\mathrm{F}}^2 = n$. If $\boldsymbol{x}$ is taken at random from the uniform distribution on $\mathbb{B}^n$, then

$$\mathbb{P}\left(1 - \epsilon \leq \frac{\|\boldsymbol{A}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \leq 1 + \epsilon\right) \geq 1 - \frac{2\|\boldsymbol{A}\boldsymbol{A}^{\vdash}\|_{\mathrm{F}}^2}{n^2\epsilon^2(2-\epsilon)^2}$$

for every $\epsilon \in (0, 1)$.

Theorem: (Böttcher, Kunis, P., 08)
If $NR \leq 1$ then

$$\mathbb{P}\left(1-\epsilon \leq \frac{\|\boldsymbol{f}\|_{\boldsymbol{w},2}}{\|f\|_2} \leq 1+\epsilon\right) \geq 1 - \frac{2(1+B_d NR)}{\mathcal{N}_d(N)\epsilon^2(2-\epsilon)^2}$$

for each $\epsilon \in (0,1)$.

$B_d = 3^{d/2}$ is Filbir/Themistoclakis constant depending only on d
$\mathcal{N}_d(N)$ is the dimension of Π_N^d
R is partition norm

$$R = \max_j \operatorname{diam} R_j := \max_j \max_{\boldsymbol{\xi},\boldsymbol{\eta}\in R_j} d(\boldsymbol{\xi},\boldsymbol{\eta})$$

Corollary:

Let $NR \leq 1$. If $0 < \alpha < d/2$, then

$$a_N := \frac{2(1+B_d NR)N^{2\alpha}}{\mathcal{N}_d(N)} \sim \frac{\Gamma(d+1)(1+B_d NR)}{N^{d-2\alpha}}$$

and

$$\mathbb{P}\left(1 - \frac{1}{N^\alpha} \leq \frac{\|\boldsymbol{f}\|_{\boldsymbol{W},2}}{\|f\|_2} \leq 1 + \frac{1}{N^\alpha}\right) \geq 1 - a_N.$$

If $0 < \beta < d$, then

$$b_N := \sqrt{\frac{2N^\beta(1+B_d NR)}{\mathcal{N}_d(N)}} \sim \frac{\sqrt{\Gamma(d+1)(1+b_d NR)}}{N^{(d-\beta)/2}}$$

and

$$\mathbb{P}\left(1 - b_N \leq \frac{\|\boldsymbol{f}\|_{\boldsymbol{W},2}}{\|f\|_2} \leq 1 + b_N\right) \geq 1 - \frac{1}{N^\beta}.$$

Theorem:

Let $d \geq 2$, $\epsilon \in (0,1)$, $\eta \in (0,1)$, $L \in (1,\infty)$, and suppose the set $\mathcal{X}$ has partition norm R and separation distance q. Then there exists a positive number $\varrho_0 = \varrho_0(d,\epsilon,\eta,L) > 0$ such that

$$\mathbb{P}\left(1-\epsilon \leq \frac{\|\boldsymbol{f}\|_{\boldsymbol{W},2}}{\|f\|_2} \leq 1+\epsilon\right) \geq 1-\eta$$

for every polynomial degree $N \geq 0$ whenever the uniformity condition $R/q < L$ and the density condition $R < \varrho_0$ hold.

Conclusions

- NFFT and inverse NFFT
- NFFT on the sphere
- Iterative reconstruction on $\mathbb{S}^2$
- Probabilistic arguments

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