On singular multiparameter eigenvalue problems

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Outline

- Two-parameter eigenvalue problem (2EP)
- Singular two-parameter eigenvalue problem
- Quadratic two-parameter eigenvalue problem (Q2EP)
- An algorithm for the extraction of the common regular part of two matrix pencils
- Examples and possible applications

Two-parameter eigenvalue problem

- Two-parameter eigenvalue problem:

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1)x &= 0\\ (A_2 + \lambda B_2 + \mu C_2)y &= 0,\end{aligned} \qquad \text{(2EP)}$$

  where A_i, B_i, C_i are $n \times n$ matrices, $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathbb{C}^n$.

- Eigenvalue: a pair (λ, μ) that satisfies (2EP) for nonzero x and y.

- Eigenvector: the tensor product $x \otimes y$.

- There are n^2 eigenvalues, which are solutions of

$$\begin{aligned}\det(A_1 + \lambda B_1 + \mu C_1) &= 0\\ \det(A_2 + \lambda B_2 + \mu C_2) &= 0.\end{aligned}$$

Tensor product approach

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1)x &= 0\\(A_2 + \lambda B_2 + \mu C_2)y &= 0\end{aligned} \tag{2EP}$$

- On $\mathbb{C}^n \otimes \mathbb{C}^n$ we define $n^2 \times n^2$ matrices

$$\begin{aligned}\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2\\\Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2\\\Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2.\end{aligned}$$

- 2EP is equivalent to a coupled GEP

$$\begin{aligned}\Delta_1 z &= \lambda \Delta_0 z\\\Delta_2 z &= \mu \Delta_0 z,\end{aligned} \tag{Δ}$$

  where $z = x \otimes y$.

- 2EP is nonsingular $\Longleftrightarrow$ Δ_0 is nonsingular

- $\Delta_0^{-1}\Delta_1$ and $\Delta_0^{-1}\Delta_2$ commute.

Numerical methods

$$\text{(2EP)} \quad \begin{array}{l}(A_1+\lambda B_1+\mu C_1)x=0\\(A_2+\lambda B_2+\mu C_2)y=0\end{array} \qquad \begin{array}{l}\Delta_0=B_1\otimes C_2-C_1\otimes B_2\\\Delta_1=C_1\otimes A_2-A_1\otimes C_2\\\Delta_2=A_1\otimes B_2-B_1\otimes A_2\end{array} \qquad \begin{array}{l}\Delta_1 z=\lambda\Delta_0 z\\\Delta_2 z=\mu\Delta_0 z\end{array} \quad (\Delta)$$

Hochstenbach, Košir, P. (2005): QZ applied to (Δ). Time complexity: $\mathcal{O}(n^6)$.

Numerical methods

$$\text{(2EP)} \quad \begin{array}{l} (A_1 + \lambda B_1 + \mu C_1)x = 0 \\ (A_2 + \lambda B_2 + \mu C_2)y = 0 \end{array} \qquad \begin{array}{l} \Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = C_1 \otimes A_2 - A_1 \otimes C_2 \\ \Delta_2 = A_1 \otimes B_2 - B_1 \otimes A_2 \end{array} \qquad \begin{array}{l} \Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z \end{array} \quad (\Delta)$$

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Algorithms that work directly with matrices A_i, B_i, C_i:

- Gradient method: Blum, Curtis, Geltner (1978), Browne, Sleeman (1982)
- Newton's method for eigenvalues: Bohte (1980)
- Generalized Rayleigh Quotient Iteration: Ji, Jiang, Lee (1992)
- Jacobi-Davidson:
  - Hochstenbach, P. (2002) for right definite 2EP,
  - Hochstenbach, Košir, P. (2005) for nonsingular 2EP,
  - Hochstenbach, P. (2008) - harmonic extraction

Singular 2EP

$$(2\text{EP}) \quad \begin{aligned}(A_1 + \lambda B_1 + \mu C_1)x &= 0\\ (A_2 + \lambda B_2 + \mu C_2)y &= 0\end{aligned} \qquad \begin{aligned}\Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2\\ \Delta_1 &= C_1 \otimes A_2 - A_1 \otimes C_2\\ \Delta_2 &= A_1 \otimes B_2 - B_1 \otimes A_2\end{aligned} \qquad \begin{aligned}\Delta_1 z &= \lambda \Delta_0 z\\ \Delta_2 z &= \mu \Delta_0 z\end{aligned} \quad (\Delta)$$

2EP is singular iff Δ_0 is singular.

For singular 2EP, there are no general results linking the eigenvalues of (2EP) and (Δ).

We know:

$$\begin{aligned}(A_1 + \lambda B_1 + \mu C_1)x &= 0\\ (A_2 + \lambda B_2 + \mu C_2)y &= 0\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\Delta_1(x \otimes y) &= \lambda \Delta_0(x \otimes y)\\ \Delta_2(x \otimes y) &= \mu \Delta_0(x \otimes y)\end{aligned}$$

Finite regular eigenvalues

A pair (λ, μ) is a finite regular eigenvalue of (2EP) if:

$$\operatorname{rank}(A_i + \lambda B_i + \mu C_i) < \max_{(s,t)\in\mathbb{C}^2} \operatorname{rank}(A_i + sB_i + tC_i)$$

for $i = 1, 2$.

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for $i = 1, 2$.

A pair (λ, μ) is a finite regular eigenvalue of matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ if:

1. $\mathrm{rank}(\Delta_1 - \lambda\Delta_0) < \max_{s\in\mathbb{C}} \mathrm{rank}(\Delta_1 - s\Delta_0)$,
2. $\mathrm{rank}(\Delta_2 - \mu\Delta_0) < \max_{t\in\mathbb{C}} \mathrm{rank}(\Delta_2 - t\Delta_0)$,
3. there exists a common eigenvector z in regular parts of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ such that

$$\begin{aligned}(\Delta_1 - \lambda\Delta_0)z &= 0,\\ (\Delta_2 - \mu\Delta_0)z &= 0.\end{aligned}$$

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2. $\text{rank}(\Delta_2 - \mu\Delta_0) < \max_{t\in\mathbb{C}} \text{rank}(\Delta_2 - t\Delta_0)$,
3. there exists a common eigenvector z in regular parts of $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ such that

$$\begin{aligned}(\Delta_1 - \lambda\Delta_0)z &= 0,\\ (\Delta_2 - \mu\Delta_0)z &= 0.\end{aligned}$$

Conjecture. Finite regular eigenvalues of (2EP) = finite regular eigenvalues of (Δ).

Quadratic 2EP

$$
\begin{aligned}
&(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x = 0 \\
&(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y = 0,
\end{aligned}
\tag{Q2EP}
$$

where $A_i, B_i, \ldots, F_i$ are $n \times n$ matrices, (λ, μ) is an eigenvalue, and $x \otimes y$ is the corresponding eigenvector. In the generic case the problem has $4n^2$ eigenvalues that are solutions of

$$
\begin{aligned}
\det(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1) &= 0 \\
\det(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2) &= 0.
\end{aligned}
$$

Jahrlebring (2008): Q2EP of a simpler form, with some of the terms $\lambda^2, \lambda\mu, \mu^2$ missing, appears in the study of linear time-delay systems for the single delay.

Linearization

$$
\begin{aligned}
(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x &= 0 \\
(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y &= 0
\end{aligned}
\qquad \text{(Q2EP)}
$$

Linearization

$$
\begin{aligned}
(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x &= 0 \\
(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y &= 0
\end{aligned}
\qquad \text{(Q2EP)}
$$

Vinnikov (1989): It follows from the theory on determinantal representations that one could write Q2EP as a two-parameter eigenvalue problem with $2n \times 2n$ matrices.

Since there is no construction this is just a theoretical result.

Linearization

$$(A_1 + \lambda B_1 + \mu C_1 + \lambda^2 D_1 + \lambda\mu E_1 + \mu^2 F_1)x = 0$$
$$(A_2 + \lambda B_2 + \mu C_2 + \lambda^2 D_2 + \lambda\mu E_2 + \mu^2 F_2)y = 0 \qquad \text{(Q2EP)}$$

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Since there is no construction this is just a theoretical result.

Best we can do is to write Q2EP as a two-parameter eigenvalue problem with $3n \times 3n$ matrices:

$$\left(\begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ \lambda x \\ \mu x \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} y \\ \lambda y \\ \mu y \end{bmatrix} = 0.$$

Weak linearization

If we multiply the matrix of the first equation

$$\begin{bmatrix} A_1 & B_1+\lambda D_1 & C_1+\lambda E_1+\mu F_1 \\ \lambda I & -I & 0 \\ \mu I & 0 & -I \end{bmatrix}$$

from left by the unimodular polynomial

$$E(\lambda,\mu)=\begin{bmatrix} I & B_1+\lambda D_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 & C_1+\lambda E_1+\mu F_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and from right by the unimodular polynomial

$$F(\lambda,\mu)=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mu I & 0 & I \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ \lambda I & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

we obtain

$$\begin{bmatrix} A_1+\lambda B_1+\mu C_1+\lambda^2 D_1+\lambda\mu E_1+\mu^2 F_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Linearization is a singular 2EP

$$A^{(1)} + \lambda B^{(1)} + \mu C^{(1)} = \begin{bmatrix} A_1 & B_1 & C_1 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_1 & E_1 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_1 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}$$

$$A^{(2)} + \lambda B^{(2)} + \mu C^{(2)} = \begin{bmatrix} A_2 & B_2 & C_2 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & D_2 & E_2 \\ I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & F_2 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{bmatrix}.$$

The matrices of the corresponding pair of generalized eigenvalue problems are

$$\begin{aligned} \Delta_0 &= B^{(1)} \otimes C^{(2)} - C^{(1)} \otimes B^{(2)}, \\ \Delta_1 &= C^{(1)} \otimes A^{(2)} - A^{(1)} \otimes C^{(2)}, \\ \Delta_2 &= A^{(1)} \otimes B^{(2)} - B^{(1)} \otimes A^{(2)}. \end{aligned}$$

Lemma. In the generic case (matrices D_1, D_2, F_1, F_2 are all nonsingular) it follows:

1. $\mathrm{rank}(\Delta_1) = \mathrm{rank}(\Delta_2) = 8n^2$,
2. $\mathrm{rank}(\Delta_0) = 6n^2$,
3. $\det(\alpha_0\Delta_0 + \alpha_1\Delta_1 + \alpha_2\Delta_2) = 0$ for all $\alpha_0, \alpha_1, \alpha_2$.

Regular eigenvalues

In the generic case (matrices D_1, D_2, F_1, F_2 are all nonsingular), we have:

1. A basis for $\ker(\Delta_1)$ is $\begin{bmatrix} 0 \\ e_i \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ e_j \\ 0 \end{bmatrix}, \quad i,j = 1,\ldots,n.$

2. A basis for $\ker(\Delta_2)$ is $\begin{bmatrix} 0 \\ D_1^{-1}E_1 e_i \\ -e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ D_2^{-1}E_2 e_j \\ -e_j \end{bmatrix}, \quad i,j = 1,\ldots,n.$

3. $\ker(\Delta_i) \subset \ker(\Delta_0)$ for $i = 1, 2$. A basis for the remaining vectors in $\ker(\Delta_0)$ is

$$\begin{bmatrix} 0 \\ D_1^{-1}(E_1 - F_1)e_i \\ -e_i \end{bmatrix} \otimes \begin{bmatrix} 0 \\ D_2^{-1}(E_2 - F_2)e_j \\ -e_j \end{bmatrix}, \quad i,j = 1,\ldots,n.$$

Theorem. The eigenvalues of Q2EP are regular eigenvalues of the coupled matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$ from the weak linearization.

Kronecker canonical structure

For a matrix pencil $A - \lambda B$ there exist nonsingular matrices P and Q such that

$$P^{-1}(A - \lambda B)Q = \widetilde{A} - \lambda \widetilde{B} = \mathrm{diag}(A_1 - \lambda B_1, \ldots, A_b - \lambda B_b)$$

is the Kronecker canonical form (KCF). Regular blocks

$$J_j(\alpha) = \begin{bmatrix} \alpha - \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha - \lambda \end{bmatrix}, \quad N_j = \begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & \ddots & -\lambda \\ & & & 1 \end{bmatrix},$$

and singular blocks

$$L_j = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix}, \quad L_j^T = \begin{bmatrix} -\lambda & & \\ 1 & \ddots & \\ & \ddots & -\lambda \\ & & 1 \end{bmatrix},$$

represent finite regular, infinite regular, right singular, and left singular blocks, respectively.

Theorem. Kronecker canonical form of pencil $\Delta_1 - \lambda \Delta_0$ (and $\Delta_2 - \mu \Delta_0$) has n^2 L_0, n^2 L_0^T, $2n^2$ N_2 blocks, and the finite regular part of size $4n^2$.

Numerical method for singular 2EP

$$\text{(2EP)} \quad \begin{array}{l}(A_1 + \lambda B_1 + \mu C_1)x = 0 \\ (A_2 + \lambda B_2 + \mu C_2)y = 0\end{array} \qquad \begin{array}{l}\Delta_0 = B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_1 = C_1 \otimes A_2 - A_1 \otimes C_2 \\ \Delta_2 = A_1 \otimes B_2 - B_1 \otimes A_2\end{array} \qquad \begin{array}{l}\Delta_1 z = \lambda \Delta_0 z \\ \Delta_2 z = \mu \Delta_0 z\end{array} \quad (\Delta)$$

We extract the common regular part of matrix pencils (Δ).

We obtain matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, and $\widetilde{\Delta}_2$ such that $\widetilde{\Delta}_0$ is nonsingular and eigenvalues of

$$\begin{array}{rcl}\widetilde{\Delta}_1 \widetilde{z} & = & \lambda \widetilde{\Delta}_0 \widetilde{z} \\ \widetilde{\Delta}_2 \widetilde{z} & = & \mu \widetilde{\Delta}_0 \widetilde{z}\end{array} \qquad (\widetilde{\Delta})$$

are common regular eigenvalues of (Δ).

For Q2EP and some other singular 2EP we can show that

regular eigenvalues of (2EP) $=$ eigenvalues of $(\widetilde{\Delta})$ $=$ regular eigenvalues of (Δ).

For the extraction of the common regular part we use the algorithm by Van Dooren (1979).

Generalized upper-triangular form

Instead of KCF, we use the generalized upper-triangular form (GUPTRI):

$$P^H(A-\lambda B)Q = \left[\begin{array}{cc|cc} A_\mu-\lambda B_\mu & & & \\ \times & A_\infty-\lambda B_\infty & & \\ \hline \times & \times & A_f-\lambda B_f & \\ \times & \times & \times & A_\epsilon-\lambda B_\epsilon \end{array}\right].$$

Pencils $A_\mu-\lambda B_\mu$, $A_\infty-\lambda B_\infty$, $A_f-\lambda B_f$, and $A_\epsilon-\lambda B_\epsilon$ contain the left singular, the infinite regular, the finite regular, and the right singular structure, respectively.

Van Dooren (1979), Demmel and Kågström (1993), software package GUPTRI.

Algorithms RRS and RLS

RRS: Extraction of the regular and the right singular part

Using SVD for the row and column compressions of the matrices A and B we find matrices P, Q with orthonormal columns such that

- $P^H(A - \lambda B)Q = \begin{bmatrix} A_f - \lambda B_f & \\ \times & A_\epsilon - \lambda B_\epsilon \end{bmatrix}$ is a regular and right singular structure of pencil $A - \lambda B$,
- the columns of Q are a basis for the eigenspace of the regular and the right singular part.

Algorithms RRS and RLS

RRS: Extraction of the regular and the right singular part

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- the columns of Q are a basis for the eigenspace of the regular and the right singular part.

RLS: Extraction of the regular and the left singular part

Using SVD for the column and row compressions of the matrices A and B we find matrices P, Q with orthonormal columns such that

- $P^H(A - \lambda B)Q = \begin{bmatrix} A_\mu - \lambda B_\mu & \\ \times & A_f - \lambda B_f \end{bmatrix}$ is a regular and left singular structure of pencil $A - \lambda B$,
- the columns of Q are a basis for the eigenspace of the regular and the left singular part.

Algorithm for the common regular part

We start with matrix pencils $\Delta_1 - \lambda\Delta_0$ and $\Delta_2 - \mu\Delta_0$, $P = I$ and $Q = I$.

1. Separate infinite and finite part.
   (a) Apply Algorithm RRS to $P^H\Delta_1 Q - \lambda P^H\Delta_0 Q$ and $P^H\Delta_2 Q - \mu P^H\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Compute Q and P with orthon. columns such that $\mathcal{Q} = \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.
   (c) If $\mathcal{Q} = \mathcal{Q}_1$ return P, Q and proceed to (2a). Otherwise, proceed to (1a).

2. Separate the finite regular part from the right singular part.
   (a) Apply Algorithm RLS on $P^H\Delta_1 Q - \lambda P^H\Delta_0 Q$ and $P^H\Delta_2 Q - \mu P^H\Delta_0 Q$. We get P_1, Q_1 and P_2, Q_2.
   (b) Compute Q and P with orthon. columns such that $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{P} = \mathcal{P}_1 \cap \mathcal{P}_2$.
   (c) If $\mathcal{P} = \mathcal{P}_1$ return P, Q and exit. Otherwise, proceed to (2a).

In the end $\widetilde{\Delta}_0 = P^H\Delta_0 Q$, $\widetilde{\Delta}_1 = P^H\Delta_1 Q$, $\widetilde{\Delta}_2 = P^H\Delta_2 Q$ and $\widetilde{\Delta}_0$ is nonsingular.

Q2EP example

$$\left(\begin{bmatrix}-3 & 4\\ 6 & -1\end{bmatrix} + \lambda\begin{bmatrix}7 & 2\\ -2 & 1\end{bmatrix} + \mu\begin{bmatrix}4 & -1\\ 9 & 4\end{bmatrix} + \lambda^2\begin{bmatrix}6 & 7\\ 5 & 2\end{bmatrix} + \lambda\mu\begin{bmatrix}10 & -3\\ 7 & 1\end{bmatrix} + \mu^2\begin{bmatrix}4 & 8\\ 6 & -3\end{bmatrix}\right) x = 0,$$

$$\left(\begin{bmatrix}-1 & 3\\ 2 & -1\end{bmatrix} + \lambda\begin{bmatrix}-1 & -4\\ 8 & 2\end{bmatrix} + \mu\begin{bmatrix}2 & 3\\ -4 & -1\end{bmatrix} + \lambda^2\begin{bmatrix}2 & 6\\ 1 & 3\end{bmatrix} + \lambda\mu\begin{bmatrix}7 & -2\\ 3 & 7\end{bmatrix} + \mu^2\begin{bmatrix}3 & -5\\ -5 & 2\end{bmatrix}\right) y = 0.$$

Matrices Δ_0, Δ_1, Δ_2 obtained in the linearization are of size 36×36.

The algorithm for the extraction of the common regular part returns matrices $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$ of size 16×16, such that $\widetilde{\Delta}_0$ is nonsingular and eigenvalues of

$$\begin{aligned}\widetilde{\Delta}_1\widetilde{z} &= \lambda\widetilde{\Delta}_0\widetilde{z}\\ \widetilde{\Delta}_2\widetilde{z} &= \mu\widetilde{\Delta}_0\widetilde{z}\end{aligned} \qquad (\widetilde{\Delta})$$

are the eigenvalue of the Q2EP.

From $(\widetilde{\Delta})$ we can compute all 16 eigenvalues of Q2EP. The largest and the smallest eigenvalue (in absolute value) truncated to 3 decimal places are $(1.799, -2.166)$ and $(0.007 \pm 0.167i, -0.507 \pm 0.1i)$.

Cubic two-parameter eigenvalue problem

$$\begin{aligned}(S_{00}+\lambda S_{10}+\mu S_{01}+\cdots+\lambda^3 S_{30}+\lambda^2\mu S_{21}+\lambda\mu^2 S_{12}+\mu^3 S_{03})x &= 0\\ (T_{00}+\lambda T_{10}+\mu T_{01}+\cdots+\lambda^3 T_{30}+\lambda^2\mu T_{21}+\lambda\mu^2 T_{12}+\mu^3 T_{03})y &= 0.\end{aligned}$$

In the general case the problem has $9n^2$ eigenvalues. A possible linearization is

$$\left(\begin{bmatrix} S_{00} & S_{10} & S_{01} & S_{20} & S_{11} & S_{02}\\ & -I & & & & \\ & & -I & & & \\ & & & -I & & \\ & & & & -I & \\ & & & & & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & S_{30} & S_{21} & S_{12}\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & S_{03}\\ 0 & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}\right)\widetilde{x} = 0$$

$$\left(\begin{bmatrix} T_{00} & T_{10} & T_{01} & T_{20} & T_{11} & T_{02}\\ & -I & & & & \\ & & -I & & & \\ & & & -I & & \\ & & & & -I & \\ & & & & & -I \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & 0 & T_{30} & T_{21} & T_{12}\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & T_{03}\\ 0 & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}\right)\widetilde{y} = 0,$$

where $\widetilde{x} = [\,1 \quad \lambda \quad \mu \quad \lambda^2 \quad \lambda\mu \quad \mu^2\,]^T \otimes x$ and $\widetilde{y} = [\,1 \quad \lambda \quad \mu \quad \lambda^2 \quad \lambda\mu \quad \mu^2\,]^T \otimes y$.

Problem is singular, rank$(\Delta_0) = 20n^2$.

Similarly we can linearize all bivariate matrix polynomials.

Model updating as a singular 2EP

Model updating (Cottin 2001, Cottin and Reetz 2006): Parameters of finite element models of multibody systems are updated to match the measured input-output data.

Updating two degrees of freedom by two measurements is equivalent to:

Find a perturbation of matrix A by a linear combination of matrices B and C, such that $A + \lambda B + \mu C$ has the prescribed eigenvalues σ_1 and σ_2.

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Updating two degrees of freedom by two measurements is equivalent to:

Find a perturbation of matrix A by a linear combination of matrices B and C, such that $A + \lambda B + \mu C$ has the prescribed eigenvalues σ_1 and σ_2.

The problem, usually treated as an optimization problem, can be expressed as a 2EP

$$\begin{aligned}(A - \sigma_1 I + \lambda B + \mu C)x &= 0,\\ (A - \sigma_2 I + \lambda B + \mu C)y &= 0.\end{aligned}$$

$\det(B \otimes C - C \otimes B) = 0$ and this is a singular 2EP.

Jacobi-Davidson method

- Jacobi–Davidson method was successfully applied to nonsingular 2EPs.
- J-D is a subspace method, in each outer step we have to solve a smaller projected 2EP.
- Now that we have a solver for singular 2EPs, we can use it in the outer step.
- This gives us a J-D for singular 2EP.

A possible application is model updating, where FEM matrices can be large and sparse, and one is interested in real and componentwise positive eigenvalues (λ, μ) that are close to the target, which is usually $(1, 1)$.

Zeros of bivariate polynomials

Suppose that we have a system of two bivariate polynomials

$$\begin{aligned} p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{ij}x^i y^j &= 0 \\ q(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} b_{ij}x^i y^j &= 0. \end{aligned}$$

Such system has n^2 solutions.

We can linearize this as a singular 2EP with matrices of size $\dfrac{n(n+1)}{2} \times \dfrac{n(n+1)}{2}$.

Matrices of 2EP are very large, but:

- We can apply Jacobi–Davidson method to compute solutions close to a target (x_0, y_0).
- Since matrices are very sparse, we need $\mathcal{O}(n^2)$ flops for one MV multiplication in J-D.
- This is the same order as for one evaluation of $p(x,y)$ and $q(x,y)$.

Conclusions

- The algorithm for the extraction of the common regular subspace of two matrix pencils.
- Q2EP can be solved via 2EP linearization.
- Possible new approach for systems of bivariate polynomials.
- Possible application in model updating.
- Ideas could be straightforward generalized to problems with three or more parameters.