Linear algebra problems arising in discontinuous Galerkin finite element discretizations

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Outline

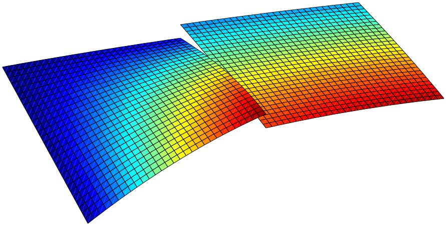
- Introduction to discontinuous Galerkin (DG) FEM
- DG-FEM for the Poisson problem
  - formulation of DG methods
  - matrix form
  - survey of the literature on solvers
- DG-FEM for Maxwell's problems
  - mixed, indefinite and eigenvalue problems

Introduction to DG-FEM

DG-FEM are finite element methods based on completely discontinuous finite element spaces

Ingredients

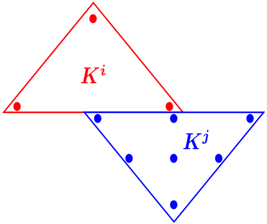
- $\mathcal{T}_h = \{K\}$ partition of the domain Ω
- $\mathcal{P}^\ell(\mathcal{T}_h)$ = piecewise polynomials of degree ℓ in each element possibly discontinuous across interelement boundaries
- Local variational formulation (element-by-element) $\longrightarrow$ interelement continuity conditions imposed within the variational formulation (no special boundary degrees of freedom, no Lagrange multipliers)

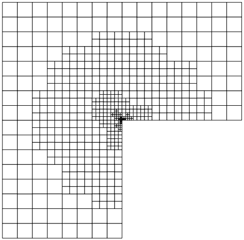


Introduction to DG-FEM

Main Features

- Wide range of PDE's treated within the same unified framework
- Flexibility in the mesh design $\rightarrow$ good for adaptivity
  - non-matching grids (*hanging nodes*)
  - non-uniform approximation degrees





- Block-diagonal (even diagonal) mass matrices
- Drawback: high number of degrees of freedom

DG for the Poisson Problem

The Poisson Problem

Given $f \in L^2(\Omega)$ and $g_{\mathcal{D}} \in H^{1/2}(\Omega)$, find $u \in H^1(\Omega)$ s.t.

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= g_{\mathcal{D}} \qquad \text{on } \partial\Omega \end{aligned}$$

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Variational Formulation

Find $u \in H^1(\Omega)$ with $u = g_\mathcal{D}$ on $\partial\Omega$ s.t.

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f\, v \qquad \forall v \in H_0^1(\Omega)$$

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- $\mathcal{T}_h$ partition of the domain; $\mathcal{F}_h$ set of all edges (faces in 3D); $\mathcal{F}_h^{\mathcal{I}}$, $\mathcal{F}_h^{\mathcal{B}}$ sets of all interior and boundary edges, resp.

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- Multiply by $v_h \in V_h$ and integrate by parts in each element

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- Sum over all elements

$$\sum_{K\in\mathcal{T}_h} \int_K \nabla u \cdot \nabla v_h - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nabla u \cdot \mathbf{n}_K \, v_h = \int_\Omega f \, v_h$$

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Key formula

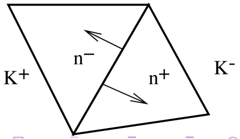
$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}\nabla u\cdot\mathbf{n}_K\, v_h = \sum_{f\in\mathcal{F}_h}\int_f \{\!\{\nabla u\}\!\}\cdot[\![v_h]\!]_N + \underbrace{\sum_{f\in\mathcal{F}_h^{\mathcal{I}}}\int_f [\![\nabla u]\!]_N\,\{\!\{v_h\}\!\}}_{=0}$$

Averages and jumps on interior edges

- $\{\!\{v\}\!\} := (v^+ + v^-)/2 \qquad \{\!\{\mathbf{q}\}\!\} := (\mathbf{q}^+ + \mathbf{q}^-)/2$
- $[\![v]\!]_N := v^+\mathbf{n}^+ + v^-\mathbf{n}^- \qquad [\![\mathbf{q}]\!]_N := \mathbf{q}^+\cdot\mathbf{n}^+ + \mathbf{q}^-\cdot\mathbf{n}^-$

Averages and jumps on boundary edges

- $\{\!\{\mathbf{q}\}\!\} = \mathbf{q}$
- $[\![v]\!]_N := v\mathbf{n},\quad [\![u]\!]_N := (u - g_{\mathcal{D}})\mathbf{n}$



DG for the Poisson Problem

DG Methods of the Interior Penalty Family (IP-DG)

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v_h - \sum_{f \in \mathcal{F}_h} \int_f \{\!\{\nabla u_h\}\!\} \cdot [\![v_h]\!]_N$$

$$= \int_\Omega f \, v_h$$

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Matrix Form

IP-DG Methods

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla u_h \cdot \nabla v_h$$

$$\underbrace{\begin{bmatrix} V & & & & & & & \end{bmatrix}}_{:=A} \mathbf{u}$$

Matrix Form

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$$\underbrace{\left[V - F - k F^T + \alpha S \right]}_{:= A} \mathbf{u}$$

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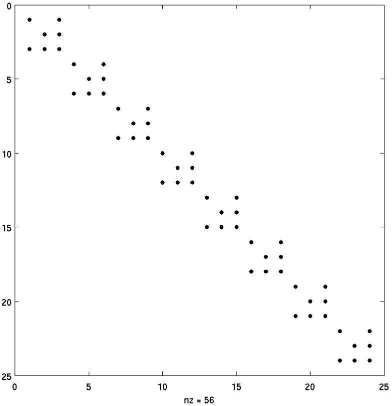
- A large, sparse, $\kappa_2(A)\sim h^{-2}$

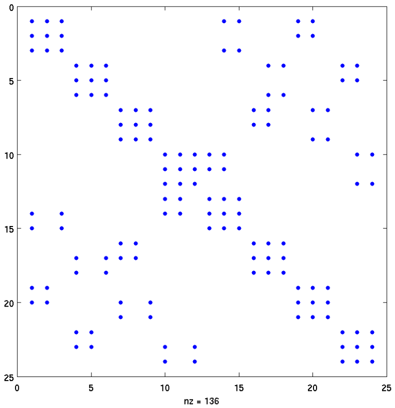
Matrix Form

V



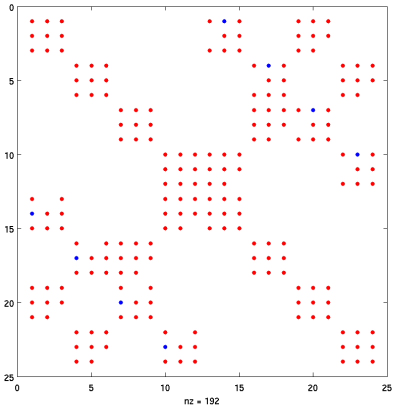
Matrix Form

$$V + \alpha S$$



Matrix Form

$$V - F - {\color{red}k}F^T + {\color{blue}\alpha}S =: A$$



Solvers

- Domain decomposition (Schwarz) preconditioners
  [Rusten, Vassilevski & Winther, 1996], [Feng & Karakashian, 2001], [Lasser & Toselli, 2003], [Brenner & Wang, 2005], [Antonietti & Ayuso, 2007-08], [Dryja, Galvis & Sarkis, 2007]
- Multigrid methods
  [Gopalakrishnan and Kanschat, 2003-04], [Brenner & Zhao, 2005], [Brenner & Owens, 2005]
- Multilevel preconditioners with cG discretization at lowest level
  [Warsa, Benzi, Wareing & Morel, 2004], [Antonietti & Ayuso, 2007]
- Norm preconditioners
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   NIP-DG provides an example in which, although the symmetric part of the operator has negative eigenvalues, the convergence of GMRES takes place all the same.

Maxwell's Problems

- Mixed problem (low-frequency approximation of the time-harmonic Maxwell's problem in insulating materials)

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \varepsilon \nabla p = \mathbf{f} \qquad & \text{in } \Omega \\ \nabla \cdot (\varepsilon \mathbf{u}) = 0 \qquad & \text{in } \Omega \\ \mathbf{n} \times \mathbf{u} = \mathbf{0}, \quad p = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

(continuity of the tangential component of $\mathbf{u}$ and continuity of p across interfaces)

Maxwell's Problems

- Mixed problem (low-frequency approximation of the time-harmonic Maxwell's problem in insulating materials)

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DG Discretizations

- Key ingredient: DG discretization of the curl-curl operator
  - similar to DG discretization of the Laplacian
    (here: penalization of the tangential jumps of $\mathbf{u}$)
  - complex vector-valued fields
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- For the mixed problem

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - \varepsilon \nabla p = \mathbf{f}$$
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  - imposition of the divergence-free constraint in a DG fashion
  - penalization of the jumps of p
  - balancing $\mathbf{V}_h$ and Q_h (polynomials of degree ℓ for $\mathbf{u}$ and $\ell + 1$ for p)
  - theoretical analysis based on an underlying stable discretization with conforming elements [Houston, Perugia & Schötzau, 2005]

Maxwell's Problems

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  - DG discretization straightforward (recall: block diagonal mass matrix)
  - theoretical analysis: same approach as for conforming discretizations, with the additional difficulty due to non-conformity [Houston, Perugia, Schneebeli & Schötzau, 2005]

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  - theoretical analysis: difficulties due to the combination of presence of a kernel and non-conformity [Buffa & Perugia, 2006]

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- $A - \omega^2 M$ non singular (indefinite), provided that ω^2 is not a discrete eigenvalue of the pencil $(A|M)$

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  - preconditioned shift-and-invert Lanczos, Jacobi-Davidson [Arbenz & Geus, 1999-2005], [Simoncini, 2003]

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- A current project: DG discretization of the Helmholtz problem

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- DG-FEM provide "complicated" test cases