UPDATING EIGENVALUES OF VIBRATING SYSTEMS

Peter Lancaster,
University of Calgary

Cortona, September 2008.

Vibrating systems

$$L(\lambda) = M\lambda^2 + D\lambda + K,$$

$M^* = M > 0,\ D^* = D,\ K^* = K.$

Vibrating systems

$$L(\lambda) = M\lambda^2 + D\lambda + K,$$

$M^* = M > 0, \; D^* = D, \; K^* = K.$

$\sigma(L) = \{\text{eigenvalues of } L\},$ (the spectrum). Assume no multiple ev.

Vibrating systems

$$L(\lambda) = M\lambda^2 + D\lambda + K,$$

$M^* = M > 0,\ D^* = D,\ K^* = K.$

$\sigma(L) = \{\text{eigenvalues of } L\}$,   (the spectrum). Assume no multiple ev.

We know a **real** ev $\lambda_0 \neq 0$ and associated evector x_0 (possibly complex), i.e. $L(\lambda_0)x_0 = 0$.

Vibrating systems

$$L(\lambda) = M\lambda^2 + D\lambda + K,$$

$M^* = M > 0, \; D^* = D, \; K^* = K.$

$\sigma(L) = \{\text{eigenvalues of } L\},$ (the spectrum). Assume no multiple ev.

We know a **real** ev $\lambda_0 \neq 0$ and associated evector x_0 (possibly complex), i.e. $L(\lambda_0)x_0 = 0$.

This real ev is to be "updated", $\lambda_0 \to \lambda_1$:

(a) Retaining symmetry of M, D, K and
(b) Without "spill-over".

The algorithm for one real ev.

Using data λ_0, x_0, compute the real number

$$x_0^* D x_0 + 2\lambda_0 (x_0^* M x_0) := \varepsilon \kappa^2 \tag{1}$$

where $\varepsilon = \pm 1$ and $\kappa > 0$.

The algorithm for one real ev.

Using data λ_0, x_0, compute the real number

$$x_0^* D x_0 + 2\lambda_0 (x_0^* M x_0) := \varepsilon \kappa^2 \tag{1}$$

where $\varepsilon = \pm 1$ and $\kappa > 0$.

ε is just the sign of this real number,
κ^2 is its absolute value.

Normalized data for this ev/evector pair is now:

$$\lambda_0, \qquad x = x_0/\kappa, \qquad \varepsilon, \tag{2}$$

and ε is the sign characteristic of the ev λ.

The algorithm for one real ev.

Form the $n \times n$ matrices of rank one:

$$S = \varepsilon \lambda_0 x x^*, \quad T = \varepsilon \lambda_0^2 x x^*, \quad U = \varepsilon \lambda_0^3 x x^*. \tag{3}$$

The algorithm for one real ev.

Form the $n \times n$ matrices of rank one:

$$S = \varepsilon \lambda_0 x x^*, \quad T = \varepsilon \lambda_0^2 x x^*, \quad U = \varepsilon \lambda_0^3 x x^*. \tag{3}$$

For update: $\lambda_0 \to \lambda_1$,

form

$$\hat{S} = \varepsilon \lambda_1 x x^*, \quad \hat{T} = \varepsilon \lambda_1^2 x x^*, \quad \hat{U} = \varepsilon \lambda_1^3 x x^*, \tag{4}$$

(and note that the sign characteristic is preserved).

Constrain $\hat{S}$ also so that $M^{-1} + (\hat{S} - S) > 0$.

Updated coefficients

Then the updated mass matrix $\hat{M}$ is given by

$$\hat{M}^{-1} = M^{-1} + (\hat{S} - S). \tag{5}$$

Updated coefficients

Then the updated mass matrix $\hat{M}$ is given by

$$\hat{M}^{-1} = M^{-1} + (\hat{S} - S). \tag{5}$$

The other updated coefficients:

$$\hat{D} = \hat{M}\left[M^{-1}DM^{-1} - (\hat{T} - T)\right]\hat{M}, \tag{6}$$

$$\hat{K} = -\hat{M}\left[M^{-1}(DM^{-1}D - K)M^{-1} + (\hat{U} - U)\right]\hat{M} + \hat{D}\hat{M}^{-1}\hat{D}. \tag{7}$$

Updating a conjugate pair

Given ev's $\mu,\ \bar{\mu}$ with evectors v_0 and w_0.

Updating a conjugate pair

Given ev's $\mu,\ \bar{\mu}$ with evectors v_0 and w_0.

Calculate the (generally complex) number

$$k = w_0^* D v_0 + 2\mu\, w_0^* M v_0. \tag{8}$$

Let κ be one of the square roots of k and form the normalized data:

$$\begin{aligned} &\text{ev} \quad \mu \quad \text{with} \quad \text{evector} \quad v = v_0/\kappa, \\ &\text{ev} \quad \bar{\mu} \quad \text{with} \quad \text{evector} \quad w = w_0/\bar{\kappa}. \end{aligned} \tag{9}$$

Updating a conjugate pair

Define matrices S, T, U of rank two:

$$S = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^*, \tag{10}$$

$$T = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix}^2 \begin{bmatrix} v & w \end{bmatrix}^*, \tag{11}$$

$$U = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \mu \\ \bar{\mu} & 0 \end{bmatrix}^3 \begin{bmatrix} v & w \end{bmatrix}^*. \tag{12}$$

Updating a conjugate pair

For updated data $\mu \to \hat{\mu}$, $\bar{\mu} \to \overline{\hat{\mu}}$,

form updated matrices $\hat{S}$, $\hat{T}$, $\hat{U}$:

$$\begin{aligned}
\hat{S} &= \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \hat{\mu} \\ \overline{\hat{\mu}} & 0 \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^*, \\
\hat{T} &= \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \hat{\mu} \\ \overline{\hat{\mu}} & 0 \end{bmatrix}^2 \begin{bmatrix} v & w \end{bmatrix}^*, \\
\hat{U} &= \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} 0 & \hat{\mu} \\ \overline{\hat{\mu}} & 0 \end{bmatrix}^3 \begin{bmatrix} v & w \end{bmatrix}^*.
\end{aligned} \tag{13}$$

Constrain $\hat{S}$ also so that $M^{-1} + (\hat{S} - S) > 0$.

Conjugate pair

As before:

$$\hat{M}^{-1} = M^{-1} + (\hat{S} - S).$$

$$\hat{D} = \hat{M}\left[M^{-1}DM^{-1} - (\hat{T} - T)\right]\hat{M},$$

$$\hat{K} = -\hat{M}\left[M^{-1}(DM^{-1}D - K)M^{-1} + (\hat{U} - U)\right]\hat{M} + \hat{D}\hat{M}^{-1}\hat{D}.$$

A general theory

$$J = \begin{bmatrix} J_c & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & \bar{J}_c \end{bmatrix} = \begin{bmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{bmatrix}.$$

A general theory

$$J = \begin{bmatrix} J_c & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & \bar{J}_c \end{bmatrix} = \begin{bmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{bmatrix}.$$

$$X = \begin{bmatrix} X_{c1} & X_{R1} & X_{R2} & X_{c2} \end{bmatrix},$$

A general theory

(X, J) form a Jordan pair for $L(\lambda)$ and, necessarily,

$$\det \begin{bmatrix} X \\ XJ \end{bmatrix} \neq 0.$$

A general theory

(X, J) form a Jordan pair for $L(\lambda)$ and, necessarily,

$$\det \begin{bmatrix} X \\ XJ \end{bmatrix} \neq 0.$$

Define

$$P := \begin{bmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{bmatrix},$$

(taking care of the sign characteristics).

N.B. $$P^* = P \text{ and } (PJ)^* = PJ.$$

A general theory

There is an X (see above) such that

$$XPX^* = 0, \qquad X(JP)X^* = M^{-1} > 0 \tag{14}$$

The moments of the system are

$$\Gamma_j = X(J^j P)X^*,$$

for all integers j for which J^j is defined.

Coefficents of $L(\lambda)$ are determined by moments (and hence X, J, P):

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2 M, \quad K = -M\Gamma_3 M + D\Gamma_1 D.$$

(First techniques are consequences of this.)

The scalar case

Example $n = 1$ and $L(\lambda) = m\lambda^2 + b\lambda + c, \quad m > 0$

(a) Real zeros $\lambda_1,\ \lambda_2$ with $\lambda_1 > \lambda_2$,

$$X = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) Non-real zeros $\mu \pm i(\lambda_1 - \lambda_2)/2$ with $\lambda_1 > \lambda_2$,

$$X = \begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \end{bmatrix}, \; J = \begin{bmatrix} \mu + i\frac{\lambda_1-\lambda_2}{2} & 0 \\ 0 & \mu - i\frac{\lambda_1-\lambda_2}{2} \end{bmatrix}, \; P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The scalar case

Example $n = 1$ and $L(\lambda) = m\lambda^2 + b\lambda + c, \quad m > 0$

(a) Real zeros $\lambda_1,\ \lambda_2$ with $\lambda_1 > \lambda_2$,

$$X = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) Non-real zeros $\mu \pm i(\lambda_1 - \lambda_2)/2$ with $\lambda_1 > \lambda_2$,

$$X = \begin{bmatrix} e^{i\pi/4} & e^{-i\pi/4} \end{bmatrix}, \; J = \begin{bmatrix} \mu + i\frac{\lambda_1 - \lambda_2}{2} & 0 \\ 0 & \mu - i\frac{\lambda_1 - \lambda_2}{2} \end{bmatrix}, \; P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In both cases, $m^{-1} = XJPX^* = \lambda_1 - \lambda_2$. Smooth transition from real to non-real eigenvalues, or vice versa, will generally induce singularities in the coefficients.

A general theory

Strategy for model updating

- Given a system with Hermitian coefficients M, D, K and $M > 0$, compute the canonical matrices X, P, J above.

Strategy for model updating

- Given a system with Hermitian coefficients M, D, K and $M > 0$, compute the canonical matrices X, P, J above.
- Make the updates in X and J to produce $\hat{X}$, $\hat{J}$ in such a way that:
  (a) P will not be disturbed, and
  (b) $\hat{X}P\hat{X}^* = 0$, and $\hat{X}(\hat{J}P)\hat{X}^* > 0$.

A general theory

Strategy for model updating

- Given a system with Hermitian coefficients M, D, K and $M > 0$, compute the canonical matrices X, P, J above.
- Make the updates in X and J to produce $\hat{X}$, $\hat{J}$ in such a way that:
  (a) P will not be disturbed, and
  (b) $\hat{X}P\hat{X}^* = 0$, and $\hat{X}(\hat{J}P)\hat{X}^* > 0$.
- Compute the moments defined by $\hat{X}$, P, $\hat{J}$, and hence new coefficients $\hat{M}$, $\hat{D}$, $\hat{K}$.

A general theory

Strategy for model updating

- Given a system with Hermitian coefficients M, D, K and $M > 0$, compute the canonical matrices X, P, J above.
- Make the updates in X and J to produce $\hat{X}$, $\hat{J}$ in such a way that:
  (a) P will not be disturbed, and
  (b) $\hat{X}P\hat{X}^* = 0$, and $\hat{X}(\hat{J}P)\hat{X}^* > 0$.
- Compute the moments defined by $\hat{X}$, P, $\hat{J}$, and hence new coefficients $\hat{M}$, $\hat{D}$, $\hat{K}$.

Can perturb ev alone, or evectors alone, or both together.

Difficulty: ensuring that $XPX^* = 0$ holds after updating.

But, this problem does not arise if the evector matrix, X, is not to be changed ($\hat{X} = X$), and adjustments are made to the ev only.

In particular: It is not possible to update one evector and retain the symmetry/no-spill-over properties.

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

$$J = \begin{bmatrix} -1.0656 + i(2.1742) & 0 & 0 & 0 \\ 0 & -0.4175 & 0 & 0 \\ 0 & 0 & -2.4513 & 0 \\ 0 & 0 & 0 & -1.0656 - i(2.1742) \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Normalised evectors:

$$X = \begin{bmatrix} -0.3021 + i(0.3333) & 0.2967 & -i(0.2201) & -0.3021 - i(0.3333) \\ -0.1567 + i(0.0774) & -0.6438 & i(0.6721) & -0.1567 - i(0.0774) \end{bmatrix}.$$

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.$$

$$J = \begin{bmatrix} -1.0656 + i(2.1742) & 0 & 0 & 0 \\ 0 & -0.4175 & 0 & 0 \\ 0 & 0 & -2.4513 & 0 \\ 0 & 0 & 0 & -1.0656 - i(2.1742) \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Normalised evectors:

$$X = \begin{bmatrix} -0.3021 + i(0.3333) & 0.2967 & -i(0.2201) & -0.3021 - i(0.3333) \\ -0.1567 + i(0.0774) & -0.6438 & i(0.6721) & -0.1567 - i(0.0774) \end{bmatrix}.$$

Updated spectrum: $-4 \pm 4i, \quad -1, \quad -4.$
Updated system:

$$\hat{M} = \begin{bmatrix} 0.5334 & -0.118 \\ & 0.6959 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 3.8308 & 0.8350 \\ & 4.0086 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} 14.8328 & 6.1635 \\ & 5.7632 \end{bmatrix}.$$

B.N. Datta, D.R. Sarkissian, Theory and computations of some inverse eigenvalue problems for the quadratic pencil, in "Structured matrices and operator theory in numerical analysis, control, signal and image processing", *Contemporary Mathematics*, 2001.

J.B. Carvalho, B.N. Datta, W. Lin, C. Wang, Symmetry preserving eigenvalue embedding in finite-element model updating of vibrating structures, *J. of Sound and Vibration*, **290**, 2006, 839-864.

M.T. Chu, S-F.Xu, Spectral decomposition of self-adjoint quadratic pencils and its applications, *Math.Comp.*, to appear.

P. Lancaster, Model updating for self-adjoint quadratic eigenvalue problems, *Lin.Alg. and its Appl*, **428**, 2008, 2778-2790.

See also: *Manchester Institute for Mathematical Sciences*: EPrint 2006.407, 2006.

Time to go!