Compact Fourier Analysis for Multigrid Methods

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Overview

1. Multigrid Method

2. Structured Matrices and Generating Functions

3. Compact Fourier Analysis for Multigrid methods based on the symbol

1. Multigrid Method

Problem: Solve linear system A x = b

Apply iterative solver like Gauss-Seidel via splitting A = M+N = (L+D)+L^T:

$$x_0, \qquad x_{k+1} = x_k + M^{-1}(b - Ax_k) = M^{-1}b + \boxed{(I - M^{-1}A)}x_k$$

Convergence depending on $||I - M^{-1}A|| = ||R||$

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Convergence depending on $||I - M^{-1}A|| = ||R||$

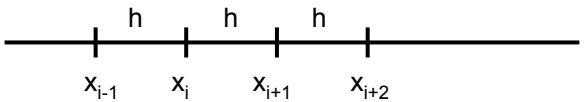
Observation: Fast convergence in eigenspace to small eigenvalues of R, resp. large eigenvalues of A

Slow convergence in eigenspace to large eigenvalues of R, resp. small eigenvalues of A

Idea: Multi-iterative method with different iterations that remove the error in different subspaces.

Laplacian

Consider elliptic PDE in 1D with discretization:



$$\boxed{\frac{d^2}{dx^2}u(x) = u_{xx} = -f(x); \quad u(x_0) = a, u(x_{n+1}) = b;}$$

$$\frac{du}{dx} \approx \frac{\Delta u}{\Delta x} = \frac{u_{i+1} - u_i}{h}$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\frac{d}{dx}u \approx \frac{\Delta}{dx}\left[\frac{u_{i+1} - u_i}{h}\right] \approx \frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2}$$

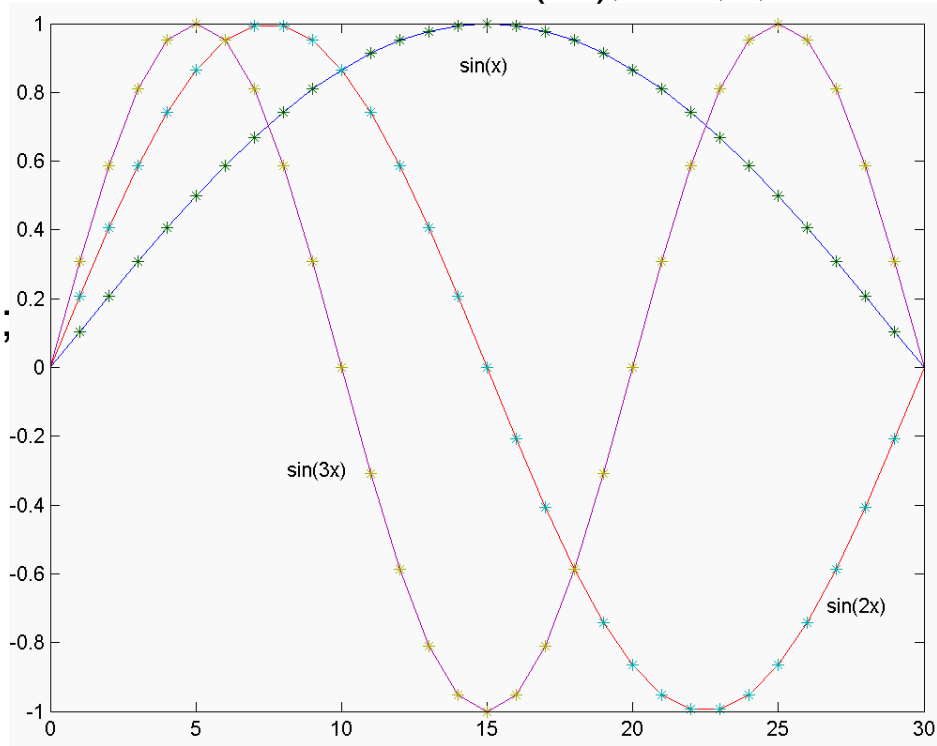
$$Au = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} -h^2 f_1 + u_0 \\ -h^2 f_2 \\ \vdots \\ -h^2 f_n + u_{n+1} \end{pmatrix} = b$$

Fourier Analysis

Eigenvectors s_k of matrix A are closely related to sine function:

$$\lambda_k = 2\left(1-\cos\left(\frac{k\pi}{n+1}\right)\right); \qquad s_k = \left(\sin\left(k\frac{j\pi}{n+1}\right)\right)_{j=1}^{n} \to \sin(kx), \qquad x \in [0,\pi]$$

Plot: sin(kx), k=1,2,3:



Low frequency:
k=1, eigenmode: sin(x),
eigenvalue: $2(1-\cos(\pi/(n+1)))\approx\pi^2/(2n^2)\approx 0$;

High frequency:
k=n, eigenmode: sin(nx),
eigenvalue: $2(1-\cos(n\,\pi/(n+1)))\approx 4$;

Fourier Analysis for Jacobi

Error reduction of m Jacobi iteration steps, considered for different eigenmodes:

$$e_m = \sum \left(\cos\left(\frac{k\pi}{n+1} \right) \right)^m \alpha_k s_k$$

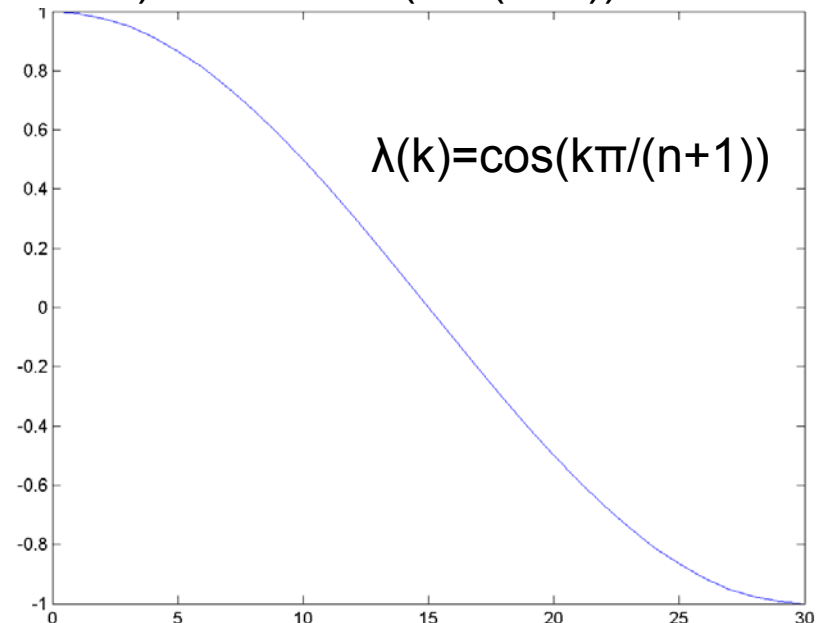
Error reduction for k=1 (low frequency component): cos(π/(n+1)) ≈ 1

Error reduction for k=n/2 (medium frequency component): cos(nπ/(2(n+1))) ≈ 0

Error reduction for k=n (high frequency component): cos(nπ/(n+1)) ≈ -1

Very good error reduction for medium frequency;

Poor error reduction for low/high frequency.



Fourier Analysis for Damped Jacobi

Is is possible to modify Jacobi iteration such that it removes high frequency error in the same way as Gauss-Seidel?

Solution: damped Jacobi:

$$x^{(k+1)} = x^{(k)} + \omega D^{-1} r_k = x^{(k)} + \omega D^{-1}(b - Ax^{(k)}) =$$

$$= \omega D^{-1} b + \left(I - \omega D^{-1} A\right) x^{(k)} = \frac{\omega}{2} b + \left(I - \frac{\omega}{2} A\right);$$

Error reduction for eigenmode:

$$\left(I - \frac{\omega}{2} A\right) s_k = \left(1 - \frac{\omega}{2} 2\left(1 - \cos\left(\frac{k\pi}{n+1}\right)\right)\right) s_k =$$

$$= \left(1 - \omega\left(1 - \cos\left(\frac{k\pi}{n+1}\right)\right)\right) s_k = \left(1 - \omega + \omega \cos\left(\frac{k\pi}{n+1}\right)\right) s_k;$$

Error Reduction damped Jacobi

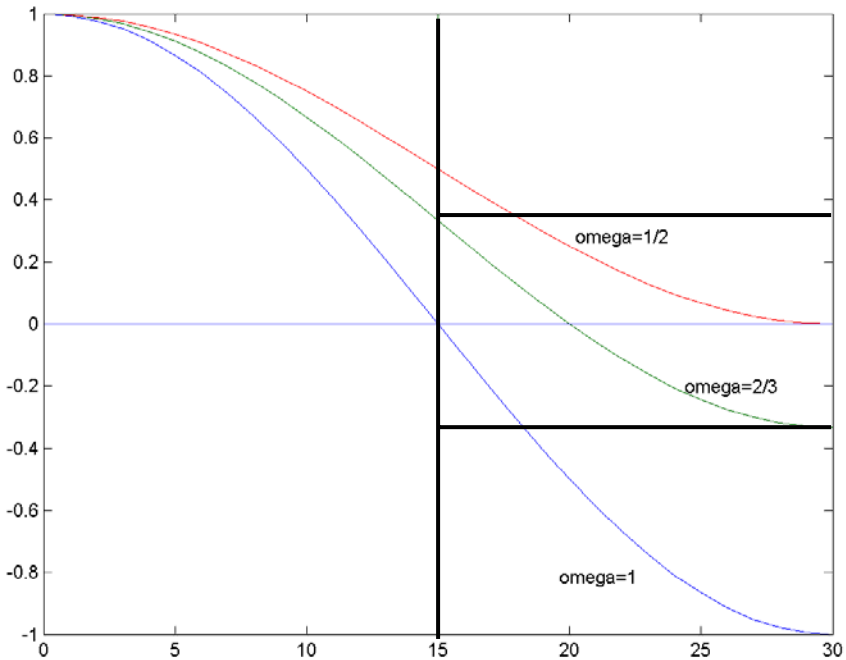
High frequency modes are related to π/2 <= x <= π, resp. n/2 <= k <= n.

k=n/2, x=π/2: $1-\omega(1-\cos(\pi/2))=1-\omega$

k=n, x=π : $1-\omega+\omega\cos(\pi)=1-2\omega$

To minimize this function, we have to choose ω such that these two values have the same absolute value but different sign: $1-\omega=2\omega-1 \Rightarrow \omega_{opt}=\frac{2}{3};$

Error reduction for these high frequency modes: $1-\omega_{opt}=1/3$.



Error reduction depending on k and omega

Coarsening

Similarly, the high frequency error components are reduced by Gauss-Seidel and Red Black – Gauss-Seidel.

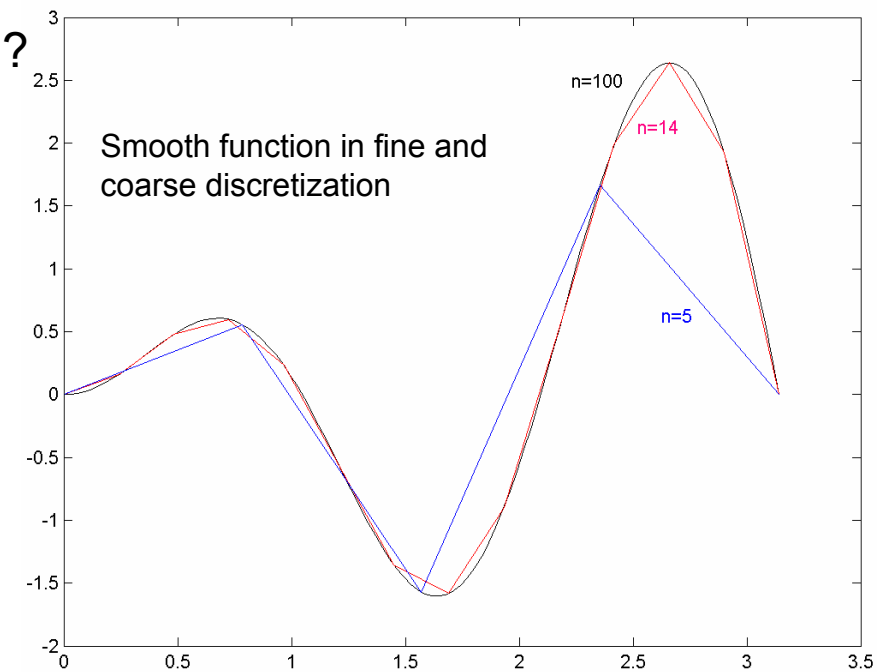
After reducing the high frequency error by some smoothing iterations with damped Jacobi or GS,
the residual $b - Ax^{(k)}$ is smooth and can be represented on a coarser grid.

Mapping from fine to coarse representation?

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \rightarrow \begin{pmatrix} u_2 \\ u_4 \\ u_6 \\ \vdots \end{pmatrix}$$

Projection or Restriction,

e.g. trivial injection



Coarsening (continued)

Better coarsening by mean value: $x_i \to \dfrac{x_{i-1} + 2x_i + x_{i+1}}{4}, i = 2,4,6,\ldots$

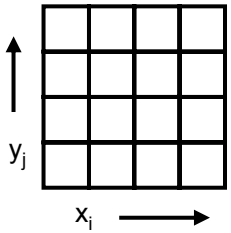
$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \to \begin{pmatrix} u_{1,coarse} \\ u_{2,coarse} \\ \vdots \\ u_{n/2,coarse} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & \\ & & 0 & 0 & 1 & 2 & 1 & \\ & & & & & & & \ddots \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = Ru$$

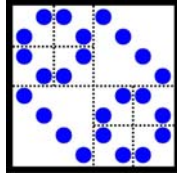
Fine to coarse via restriction R

Coarse to fine via prolongation P, e.g.

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & \\ & & \ddots \end{pmatrix}\begin{pmatrix} u_{1,coarse} \\ u_{2,coarse} \\ \vdots \\ u_{n/2,coarse} \end{pmatrix} = Pu_{coarse} = R^T u_{coarse}; \qquad u_i = \begin{cases} u_i & for \quad i \quad odd \\ (u_i + u_{i+1})/2 & for \quad i \quad even \end{cases}$$

PDE in 2D





$$u(x,y):\ u_{xx}+u_{yy}=-f(x,y);\quad u(x,y)=g(x,y)\quad for\quad x_0=a, x_{n+1}=b, y_0=a, y_{n+1}=b$$

$$\frac{-u_{i+1,j}+2u_{i,j}-u_{i-1,j}}{h^2}+\frac{-u_{i,j+1}+2u_{i,j}-u_{i,j-1}}{h^2}=-f_{i,j} \longrightarrow$$

$$A=A_1\otimes I+I\otimes A_1=\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_1 \end{pmatrix}+\begin{pmatrix} 2I & -I & \\ -I & 2I & \ddots \\ & \ddots & \ddots \end{pmatrix}=$$

$$\left(\begin{array}{ccc|ccc|c} 4 & -1 & & -1 & & & \\ -1 & 4 & \ddots & & -1 & & \\ & \ddots & \ddots & & & \ddots & \\ \hline -1 & & & 4 & -1 & & \ddots \\ & -1 & & -1 & 4 & \ddots & \\ & & \ddots & & \ddots & \ddots & \ddots \\ \hline & & & \ddots & & \ddots & \ddots \end{array}\right)$$

Twogrid and Multigrid

(1) Apply a few steps smoothing iterations (damped Jacobi, GS, RB-GS)

(2) Coarse grid correction:

- Consider the residual equation $A(x^{(k)}+x_{correction})=b$, that is related to the best approximate solution $x^{(k)}$:

$$Ax_{correction} = b-Ax^{(k)} = r_k$$

- Restriction to the coarse grid via R
- Solve Residual equation on coarse grid
- Prolongate the solution back to the fine grid
- Add the correction to $x^{(k)}$

Repeat until convergence.

V - Cycle

Multigrid:

Presmoothing
Coarsening
Presmoothing
Coarsening
Presmoothing
Coarsening
Solve coarse system
Prolongation
Postsmoothing
Prolongation
Postsmoothing
Prolongation
Postsmoothing

If coarse system is „similar“ to original problem!

Two Grid Error Reduction

Smoothing with M leads to error reduction by $I - M^{-1}A$

Error reduction by Coarse Grid Correction: $x_{new} = x_{old} + PA_{coarse}^{-1}P^T\left(b - Ax_{old}\right)$

$$e_{new} = x_{new} - \bar{x} = x_{old} + PA_{coarse}^{-1}P^T(b - Ax_{old}) - \left(\bar{x} + PA_{coarse}^{-1}P^T(b - A\bar{x})\right) =$$
$$= e_{old} - \left(PA_{coarse}^{-1}P^T\right)Ae_{old} = \left(I - \left(PA_{coarse}^{-1}P^T\right)A\right)e_{old}$$

Coarse matrix A_{coarse} is given as Galerkin projection $A_{coarse} = P^T AP = RAR^T$

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Overall error reduction by pre/post-smoothing by m_{pre}, resp. m_{post} iterations steps with M_{pre}, resp. M_{post}, and Coarse Grid Correction:

$$\boxed{\left(I - M_{post}^{-1}A\right)^{m_{post}} \cdot \left(I - P\left(P^T AP\right)^{-1}P^T \cdot A\right) \cdot \left(I - M_{pre}^{-1}A\right)^{m_{pre}}}$$

2. Structured Matrices and Generating Functions

Connection [matrix ←→ function] for the class of circulant matrices:

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & & \vdots \\ \vdots & & \ddots & c_1 \\ c_1 & \cdots & c_{n-1} & c_0 \end{pmatrix} = F^H \Lambda F, \qquad \Lambda_k = p(\omega^k)$$

$$p(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}; \qquad x = e^{i\varphi}; \qquad \omega = \exp\left(\frac{2\pi i}{n}\right);$$

$$spectrum(C) \subseteq range(C) \subseteq range(p)$$ on the unit circle.

Toeplitz Matrices and Functions

Toeplitz matrix $T_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix} = T_n(t_{n-1},\ldots,t_1,t_0,t_{-1},\ldots,t_{-n+1}) = T_n(f);$

Generating function $f(x) = \sum_{j=-\infty}^{\infty} t_j e^{-ijx}$ or symbol as function on the unit circle.

$$spectrum(T_n(f)) \subseteq range(T_n(f)) \subseteq range(f)$$

Example: T_n=tridiag(-1,2,-1) with function f(x) = -exp(ix) + 2 - exp(-ix) = 2(1-cos(x))

Band (Sparse) Toeplitz Matrices

$T_n(f)$ and $T_n(g)$ banded matrices, resp. f(x) and g(x) trigonometric polynomials:

$$T_n(f)\cdot T_n(g) = T_n(f\cdot g) + low\_rank = T_n(g)\cdot T_n(f) + low\_rank$$

$$T_n^{-1}(f)\cdot T_n(g) = T_n\left(\frac{g}{f}\right) + low\_rank \qquad for \quad \frac{g}{f} \quad continuous$$

For general spd Toeplitz matrices:

$$spectrum\left(T_n^{-1}(f)T_n(g)\right) \subseteq range\left(T_n^{-1}(f)T_n(g)\right) \subseteq range\left(\frac{g}{f}\right)$$

Proof: $\lambda \notin range\left(\frac{g}{f}\right) \Leftrightarrow \frac{g}{f} > \lambda (or \quad < \lambda) \forall x \Leftrightarrow g - \lambda f > 0 (< 0) \forall x \Leftrightarrow$

$$\Leftrightarrow T_n(g - \lambda f) > 0 (< 0) \Rightarrow \lambda \notin spectrum\left(T_n(g) = \lambda T_n(f)\right) \Leftrightarrow$$

$$\Leftrightarrow \lambda \notin spectrum\left(T_n^{-1}(f)T_n(g)\right)$$

Block and Multilevel Toeplitz

Symbol for multilevel Toeplitz: $\begin{pmatrix} T_0 & T_{-1} & \cdots & T_{-n} \\ T_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{-1} \\ T_n & \cdots & T_1 & T_0 \end{pmatrix}, \quad T_j \quad Toeplitz$

Symbol as scalar function in variables $f(x,y) = \sum_{k,j} t_{kj} e^{-ikx} e^{-ijy}$

Symbol for Block Toeplitz: $\begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{n-1,n} \\ T_{n,1} & \cdots & T_{n,n-1} & T_{n,n} \end{pmatrix}, \quad T_{j,k} \quad Toeplitz \quad matrices$

Symbol as block function $F(x) = \begin{pmatrix} f_{11}(x) & \cdots \\ \vdots & \ddots \end{pmatrix}$

Multilevel Block Toelitz: symbol as matrix functions in more variables F(x,y,..)

Block and Multilevel Toeplitz

Symbol as block function

$$F(x) = \begin{pmatrix} f_{11}(x) & \cdots \\ \vdots & \ddots \end{pmatrix}$$

$$\begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{n-1,n} \\ T_{n,1} & \cdots & T_{n,n-1} & T_{n,n} \end{pmatrix}, \quad \longleftrightarrow \quad \begin{pmatrix} T_0 & T_{-1} & \cdots & T_{1-n} \\ T_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{-1} \\ T_{n-1} & \cdots & T_1 & T_0 \end{pmatrix},$$

with $T_{j,k}$ *Toeplitz matrices* *with* T_j *general matrices*

3. Multigrid by symbol

Earlier work on Multigrid for structured matrices: R. Chan, S. Serra,…

Replace in each step of Multigrid the matrices by their block symbols:

- original matrix
- smoother
- restriction/prolongation

Use block symbols for
- smoothing analysis
- analysis of the overall error reduction of Twogrid
- design of Multigrid (projection, smoother)

Fine/Coarse by Block Function

Goal: Write Twogrid step in symbol → Fourier Analysis

In Multigrid we have to deal with two classes of grid point:
- grid points that appear also on the coarse level, and
- grid points that are only fine, but non-coarse

To model these two classes of entries in our vector f, resp. Matrix A, we have to introduce Block Symbols or Block generating functions.

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ & & & & \end{pmatrix} \to -e^{ix} + 2 - e^{-ix};$$

Scalar case

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \\ & & & & \ddots \end{pmatrix} \to -e^{ix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + e^{-ix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1-e^{ix} \\ -1-e^{-ix} & 2 \end{pmatrix}$$

Block case

Fine/Coarse in Block Matrix

$$
\begin{array}{c}
\begin{array}{cccc} \text{coarse} & \text{noncoarse} & \text{coarse} & \text{nonc.} \;\; \ldots \end{array} \\
\begin{pmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
 & -1 & 2 & -1 \\
 & & -1 & 2 \\
 & & & & \ddots
\end{pmatrix}
\end{array}
\begin{array}{l}
\leftarrow \text{coarse} \\
\leftarrow \text{noncoarse} \\
\leftarrow \text{coarse} \\
\leftarrow \text{noncoarse} \\
\vdots
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cc} \text{coarse} & \text{noncoarse} \end{array} \\
\begin{pmatrix}
2 & -1-e^{ix} \\
-1-e^{-ix} & 2
\end{pmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{cc} \text{coarse} & \text{noncoarse} \end{array} \\
\begin{pmatrix}
2 & -\alpha \\
-\overline{\alpha} & 2
\end{pmatrix}
\end{array}
$$

Coarse/noncoarse ←→ odd/even

Therefore, the projection from fine to coarse is given by picking every second row/column in the full matrix,
resp. picking the first row/column in the symbol.

Trivial injection:

$$
E = \begin{pmatrix}
1 & 0 & & & & \\
 & 0 & 1 & 0 & & \\
 & & & 0 & 1 & \\
 & & & & & \ddots
\end{pmatrix}
\leftrightarrow \begin{pmatrix} 1 & 0 \end{pmatrix}
$$

Fine to Coarse Reduction

Galerkin reduction via trivial injection: $\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -\alpha \\ -\overline{\alpha} & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2$

Better projection by (1 2 1) stencil taking the mean value of neighbouring points. The related projection matrix R can be seen as combination of full (1 2 1) stencil matrix, restricted by trivial injection in the form

$$R = \begin{pmatrix} 1 & 2 & 1 & & & \\ & & 1 & 2 & 1 & \\ & & & & & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & & & \\ & & 1 & 0 & 0 & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & & \ddots \\ & \ddots & & \ddots \end{pmatrix} = E \cdot B$$

with n x n – Toeplitz matrix B = tridiag(1,2,1) = $T_n(2,1,0,\ldots,0)$

Fine to Coarse Reduction

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with n x n – Toeplitz matrix B = tridiag(1,2,1) = $T_n(2,1,0,\ldots,0)$

Symbol for B: $b(x) = 2(1+\cos(x))$; $\quad B(x) = \begin{pmatrix} 2 & \alpha \\ \overline{\alpha} & 2 \end{pmatrix}$;

$$A_{coarse} = RAR^T = (EB)A(BE^T);$$

$$f_{coarse}(x) = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}2 & \alpha \\ \overline{\alpha} & 2\end{pmatrix}\begin{pmatrix}2 & -\alpha \\ -\overline{\alpha} & 2\end{pmatrix}\begin{pmatrix}2 & \alpha \\ \overline{\alpha} & 2\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} =$$

$$= 2\left(4 - |\alpha|^2\right) = 2\left(4 - (2 + 2\cos(x))\right) = 2(2 - \cos(x)) = 2f(x);$$

Smoother ←→ Symbol

Jacobi: $M = diag(A) = 2I \quad \rightarrow \quad m(x) = 2 \quad and \quad M(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

GS: $M = \begin{pmatrix} 2 & & & \\ -1 & 2 & & \\ & \ddots & \ddots & \\ & & -1 & 2 \end{pmatrix} \quad \rightarrow \quad m(x) = 2 - e^{ix} \quad and \quad M(x) = \begin{pmatrix} 2 & -e^{ix} \\ -1 & 2 \end{pmatrix};$

Now all components can be written as block symbols:
The matrix A itself (scalar and block function)
The smoother M (scalar and block function)
The restriction R (block function and trivial injection)
The coarse system (scalar function)

Hence we can analyse the smoother and the overall error in terms of block functions

Smoothing analysis via scalar symbol

Jacobi: $$I - M^{-1}A = I - \omega D^{-1}A \rightarrow e(x) = 1 - \omega \frac{1}{2} 2(1-\cos(x)) =$$

$$= 1 - \omega + \omega \cos(x), \qquad x \in \left[\frac{\pi}{2}, \pi\right]$$

e(x) monotonic **function**,

1-ω and 1-2ω optimal for ω_{opt}=2/3, e(ω)<=1/3 .

GS: $$I - M^{-1}A = I - \omega L^{-1}A \rightarrow e(x) = 1 - \omega \frac{2(1-\cos(x))}{2-\exp(ix)}$$

$$\left.\begin{array}{l} x = \frac{\pi}{2} : \left|1 - \omega \frac{2}{2-i}\right| = \sqrt{\frac{5 - 8\omega + 4\omega^2}{5}} \xrightarrow{\omega=1} 1/\sqrt{5} \\ x = \pi : \left|1 - \omega \frac{4}{3}\right| = \left|\frac{3-4\omega}{3}\right| \xrightarrow{\omega=1} \left|-\frac{1}{3}\right| = 1/3 \end{array}\right\} \quad e(x) \leq 1/\sqrt{5}$$

Symmetric GS: $(I - \omega L^{-1}A)(I - \omega L^{-T}A)$ ω=1 → e <= 1/5

Smoothing analysis via Block Function

$$F(x) = \begin{pmatrix} 2 & -\alpha \\ -\overline{\alpha} & 2 \end{pmatrix}$$ block symbol of matrix A

Eigenvalues of F: $\lambda_1 = 2 + |\alpha| = 2 + |1 + e^{ix}| \geq 2;$

$$\lambda_2 = 2 - |\alpha| = 2 - |1 + e^{ix}| \leq 2, \qquad \lambda_2 = 0 \qquad for \quad x = 0;$$

Smoothing is related to eigenvalues >= 2, hence to λ_1 with eigenvector

$$u_1 = \frac{1}{\sqrt{2}|\alpha|} \begin{pmatrix} \alpha \\ -|\alpha| \end{pmatrix}$$

High frequency components are related to u_1 for x in full [0,π].

Therefore, determining the behaviour on high frequency components can be achieved by the projection $u_1^H F u_1$.

Advantage of blockwise analysis: Take into account different character of grid points!

Smoothing analysis for Jacobi

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\omega}{2}\begin{pmatrix} 2 & -\alpha \\ -\overline{\alpha} & 2 \end{pmatrix} = \begin{pmatrix} 1-\omega & \omega\alpha/2 \\ \omega\overline{\alpha}/2 & 1-\omega \end{pmatrix}$$

Projection on high frequency components:

$$u_1^H \begin{pmatrix} 1-\omega & \omega\alpha/2 \\ \omega\overline{\alpha}/2 & 1-\omega \end{pmatrix} u_1 = \frac{1}{2|\alpha|^2}\begin{pmatrix} \overline{\alpha} & -|\alpha| \end{pmatrix}\begin{pmatrix} 1-\omega & \omega\alpha/2 \\ \omega\overline{\alpha}/2 & 1-\omega \end{pmatrix}\begin{pmatrix} \alpha \\ -|\alpha| \end{pmatrix} =$$

$$= \frac{1}{2|\alpha|^2}\left(2(1-\omega)|\alpha|^2 - \omega|\alpha|^3\right) = 1-\omega-\omega\frac{|\alpha|}{2}$$

$$|\alpha| = \left|1+e^{ix}\right| \in [0,2\pi]; \qquad \begin{matrix} \alpha = 0 : 1-\omega \\ \alpha = 2 : 1-2\omega \end{matrix} \rightarrow \omega_{opt} = \frac{2}{3}$$

The same result as in the scalar smoothing analysis!

Smoothing analysis for GS and RB-GS

GS:

$$u_1^H\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\omega\begin{pmatrix}2 & -e^{ix}\\ -1 & 2\end{pmatrix}^{-1}\begin{pmatrix}2 & -\alpha\\ -\overline{\alpha} & 2\end{pmatrix}\right]u_1, \qquad \alpha = 1+e^{ix}$$

RB-GS:

$$A\rightarrow\begin{pmatrix}2 & & & -1 & & \\ & \ddots & & -1 & \ddots & \\ & & 2 & & \ddots & -1\\ -1 & -1 & & 2 & & \\ & \ddots & \ddots & & \ddots & \\ & & -1 & & & 2\end{pmatrix}; \qquad L=\begin{pmatrix}2 & & & & & \\ & \ddots & & & & \\ & & 2 & & & \\ -1 & -1 & & 2 & & \\ & \ddots & \ddots & & \ddots & \\ & & -1 & & & 2\end{pmatrix}\Rightarrow\begin{pmatrix}2 & 0\\ -\overline{\alpha} & 2\end{pmatrix}$$

$$u_1^H\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\omega\begin{pmatrix}2 & 0\\ -\overline{\alpha} & 2\end{pmatrix}^{-1}\begin{pmatrix}2 & -\alpha\\ -\overline{\alpha} & 2\end{pmatrix}\right]u_1 =$$

$$= u_1^H\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\omega\begin{pmatrix}1/2 & 0\\ -\overline{\alpha}/4 & 1/2\end{pmatrix}\begin{pmatrix}2 & -\alpha\\ -\overline{\alpha} & 2\end{pmatrix}\right]u_1$$

Analysis of Coarse Grid Correction CGC

Block symbol of the Coarse Grid Correction CGC:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 \\ \overline{\alpha} \end{pmatrix} \frac{1}{4(1-\cos(x))} \begin{pmatrix} 2 & \alpha \end{pmatrix} \begin{pmatrix} 2 & -\alpha \\ -\overline{\alpha} & 2 \end{pmatrix} =$$

$$= \frac{1}{4(1-\cos(x))} \left[\begin{pmatrix} 4-4\cos(x) & 0 \\ 0 & 4-4\cos(x) \end{pmatrix} - \begin{pmatrix} 8-2|\alpha|^2 & 0 \\ 4\overline{\alpha} - |\alpha|^2 \overline{\alpha} & 0 \end{pmatrix} \right] =$$

$$= \frac{1}{4(1-\cos(x))} \begin{pmatrix} 0 & 0 \\ \overline{\alpha}(2-2\cos(x)) & 4-4\cos(x) \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 \\ \overline{\alpha}/2 & 1 \end{pmatrix}}$$

Overall Error Analysis

Red-Black postsmoothing:

$$I - M^{-1}A = M^{-1}(M - A) \rightarrow M^{-1}\left[\begin{pmatrix} 2 & -\alpha \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -\alpha \\ -\overline{\alpha} & 2 \end{pmatrix}\right] = M^{-1}\begin{pmatrix} 0 & 0 \\ \overline{\alpha} & 0 \end{pmatrix}$$

Postsmoothing and CGC:

$$\left(I - M^{-1}A\right)\left(I - PA_c^{-1}P^T A\right) \rightarrow M^{-1}\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \overline{\alpha}/2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, standard projection with RB-GS gives MG as a direct solver = Twogrid leads to error 0 after one step and the coarse system is the original A. Therefore in the V-cycle only one step of smoothing is necessary and only one V-cycle run.

Short view on 2D

Block Matrix A → block matrix function F(x,y), e.g. of size k x k

Projection part B → block matrix function B(x,y)

Trivial injection → block matrix function p=(1 0 ..0)

Pojection P → P(x,y) = B(x,y) p^T

Coarse Grid matrix → scalar function f(x,y) = p B(x,y)F(x,y)B(x,y) p^T = PFP^T

Smoother M → block matrix function M(x,y)

$$\left(I - M_{post}^{-1} A\right)^{m_{post}} \cdot \left(I - P\left(P^T AP\right)^{-1} P^T \cdot A\right) \cdot \left(I - M_{pre}^{-1} A\right)^{m_{pre}}$$

Example

Matrix: $A_1 = tridiag(-1,4,-1); \qquad A = tridiag(-I, A_1, -I)$

Symbol: $$F(x,y) = \left(\begin{array}{cc|cc} 4 & -\alpha & -\beta & 0 \\ -\bar{\alpha} & 4 & 0 & -\beta \\ \hline -\bar{\beta} & 0 & 4 & -\alpha \\ 0 & -\bar{\beta} & -\bar{\alpha} & 4 \end{array}\right)$$ with $\alpha = 1 + e^{ix}, \qquad \beta = 1 + e^{iy},$

GS-Smoother: $$M_{GS} = \begin{pmatrix} 4 & -e^{ix} & -e^{iy} & 0 \\ -1 & 4 & 0 & -e^{iy} \\ -1 & 0 & 4 & -e^{ix} \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

Error

$$\boxed{\left(I - M^{-1}A\right)\cdot\left(I - P\left(P^T AP\right)^{-1}P^T \cdot A\right)}$$

$$\left(I - M^{-1}F\right)\cdot CGC = M^{-1}\left(M - F\right)\cdot CGC$$

CGC-symbol is singular of rank k-1 by construction:

$$CGC = \begin{pmatrix} d_1 & \cdots & d_{k-1} \end{pmatrix}\cdot\begin{pmatrix} c_1 \\ \vdots \\ c_{k-1} \end{pmatrix}$$

Find smoother M such that: $\left(M - F\right)\cdot\begin{pmatrix} d_1 & \cdots & d_{k-1} \end{pmatrix} \overset{!}{=} 0$

M – F of rank k-1, if possible.

Example:

$$F = \begin{pmatrix} f_{11} & f_1^T \\ f_1 & F_{11} \end{pmatrix} \qquad CGC = \begin{pmatrix} 0 & \cdots & 0 \\ * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{pmatrix}$$

$$M = \begin{pmatrix} f_{11} & f_1^T \\ 0 & F_{11} \end{pmatrix} \qquad M - F = \begin{pmatrix} 0 & 0 \\ f_1 & 0 \end{pmatrix}$$

$$\left(I - M^{-1}F\right)\cdot CGC = M^{-1}\cdot(M-F)\cdot CGC = M^{-1}\cdot\begin{pmatrix} 0 & 0 \\ f_1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix} = 0$$

Conclusions

Compact Fourier analysis based on the symbol

- can lead to Multigrid as a direct solver
- helps to analyse smoothing and overall error taking into account the different character of grid points
- helps to design efficient MG in view of overall error:
  optimal for fixed projection,
  optimal for fixed smoother,
  optimal combination of projection and smoother.