Cortona 15–19 September 2008

Finiteness properties of sets of matrices

Nicola Guglielmi, University of L'Aquila

joint research with

Antonio Cicone (University of L'Aquila)
Stefano Serra Capizzano (University of Insubria)
Marino Zennaro (University of Trieste)

Joint spectral radius

In a time varying discrete linear dynamical system, the maximal asymptotic rate of growth of the trajectories is determined by the

joint spectral radius

of the associated family of matrices.

In particular, the **j.s.r.** characterizes also the stability properties.

Summary

- Generalizations of the spectral radius.

Summary

- Generalizations of the spectral radius.
- Theoretical framework.

Summary

- Generalizations of the spectral radius.
- Theoretical framework.
- Finiteness results. Algorithms.

Summary

- Generalizations of the spectral radius.
- Theoretical framework.
- Finiteness results. Algorithms.

Some motivations

(1) Stability of numerical methods for differential equations.

(2) Robust control.

(3) Wavelets.

(4) Capacity of codes with forbidden patterns.

(5) Consensus algorithms.

Uniform asymptotic stability (u.a.s.)

Consider the discrete linear time dependent dynamical system

$$y_{t+1} = X_t\, y_t, \qquad t = 0, 1, 2, \ldots$$

where $y_0 \in \mathbf{R}^k$ and $X_t \in \mathbf{R}^{k,k}$ is an arbitrary element of

$$\mathcal{F} = \{A_i\}_{i \in \mathcal{I}} \quad (\mathcal{I} \text{ finite set of indexes})$$

Uniform asymptotic stability (u.a.s.)

Consider the discrete linear time dependent dynamical system

$$y_{t+1} \;=\; X_t\, y_t, \qquad t = 0, 1, 2, \ldots$$

where $y_0 \in \mathbf{R}^k$ and $X_t \in \mathbf{R}^{k,k}$ is an arbitrary element of

$$\mathcal{F} = \{A_i\}_{i \in \mathcal{I}} \qquad (\mathcal{I} \text{ finite set of indexes})$$

U.a.s. means that $\lim_{t\to\infty} y_t = 0 \;\; \forall\, y_0$ or equivalently that the set

$$\Sigma_n(\mathcal{F}) \;=\; \bigcup_{i_1,\ldots,i_n \in \mathcal{I}} A_{i_n} \cdots A_{i_1}$$

of all products of length n vanishes as $n \to \infty$.

Uniform asymptotic stability (u.a.s.)

Consider the discrete linear time dependent dynamical system

$$y_{t+1} = X_t \, y_t, \qquad t = 0, 1, 2, \ldots$$

where $y_0 \in \mathbf{R}^k$ and $X_t \in \mathbf{R}^{k,k}$ is an arbitrary element of

$$\mathcal{F} = \{A_i\}_{i \in \mathcal{I}} \quad (\mathcal{I} \text{ finite set of indexes})$$

U.a.s. means that $\lim_{t \to \infty} y_t = 0 \ \ \forall \, y_0$ or equivalently that the set

$$\Sigma_n(\mathcal{F}) = \bigcup_{i_1, \ldots, i_n \in \mathcal{I}} A_{i_n} \cdots A_{i_1}$$

of all products of length n vanishes as $n \to \infty$.

For a single matrix $\Sigma_n(A) = A^n$. Hence u.a.s. $\Longleftrightarrow \rho(A) < 1$.

Generalizations of the spectral radius

(1) Joint spectral radius (Rota & Strang (1960)):

Set $\hat{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \|P\|^{1/n}$ and define $\hat{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \hat{\rho}_n$.

Generalizations of the spectral radius

(1) Joint spectral radius (Rota & Strang (1960)):

Set $\hat{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \|P\|^{1/n}$ and define $\hat{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \hat{\rho}_n$.

(2) Generalized spectral radius (Daubechies *et al.* (1992)):

Set $\bar{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \rho(P)^{1/n}$ and define $\bar{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \bar{\rho}_n$.

Generalizations of the spectral radius

(1) Joint spectral radius (Rota & Strang (1960)):

Set $\hat{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \|P\|^{1/n}$ and define $\hat{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \hat{\rho}_n$.

(2) Generalized spectral radius (Daubechies *et al.* (1992)):

Set $\bar{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \rho(P)^{1/n}$ and define $\bar{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \bar{\rho}_n$.

General results (Berger & Wang (1992)).

$$\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F}) \quad =: \rho(\mathcal{F}),$$

Generalizations of the spectral radius

(1) Joint spectral radius (Rota & Strang (1960)):

Set $\hat{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \|P\|^{1/n}$ and define $\hat{\rho}(\mathcal{F}) = \limsup_{n \to \infty} \hat{\rho}_n$.

(2) Generalized spectral radius (Daubechies *et al.* (1992)):

Set $\bar{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \rho(P)^{1/n}$ and define $\bar{\rho}(\mathcal{F}) = \limsup_{n \to \infty} \bar{\rho}_n$.

General results (Berger & Wang (1992)).

$$\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F}) \quad =: \rho(\mathcal{F}), \qquad \text{u.a.s} \iff \rho(\mathcal{F}) < 1.$$

Generalizations of the spectral radius

(1) Joint spectral radius (Rota & Strang (1960)):

Set $\hat{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \|P\|^{1/n}$ and define $\hat{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \hat{\rho}_n$.

(2) Generalized spectral radius (Daubechies *et al.* (1992)):

Set $\bar{\rho}_n = \max_{P \in \Sigma_n(\mathcal{F})} \rho(P)^{1/n}$ and define $\bar{\rho}(\mathcal{F}) = \limsup_{n\to\infty} \bar{\rho}_n$.

General results (Berger & Wang (1992)).

$$\hat{\rho}(\mathcal{F}) = \bar{\rho}(\mathcal{F}) \quad =: \rho(\mathcal{F}), \qquad\qquad \text{u.a.s} \iff \rho(\mathcal{F}) < 1.$$

Simple estimates for $\rho(\mathcal{F})$: (Daubechies & Lagarias (1992))

$$\bar{\rho}_n \leq \rho(\mathcal{F}) \leq \hat{\rho}_n \qquad \forall n \geq 1.$$

Extremal norms

A further generalization (Elsner (1995)):

$$\rho(\mathcal{F}) = \inf_{\|\cdot\| \in \mathcal{N}} \|\mathcal{F}\| \quad \text{with } \|\mathcal{F}\| = \max_{A \in \mathcal{F}} \|A\|;$$

where $\mathcal{N}$ is the set of operator norms ($\Longrightarrow \rho(\mathcal{F}) \leq \|\mathcal{F}\|$).

Extremal norms

A further generalization (Elsner (1995)):

$$\rho(\mathcal{F}) \;=\; \inf_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\| \quad \textbf{with } \|\mathcal{F}\| = \max_{A\in\mathcal{F}} \|A\|;$$

where $\mathcal{N}$ is the set of operator norms ($\Longrightarrow \; \rho(\mathcal{F}) \le \|\mathcal{F}\|$).

When the inf is a min the family is said to be non-defective,

$$\|\cdot\|_\star \;\longrightarrow\; \min_{\|\cdot\|\in\mathcal{N}} \|\mathcal{F}\| \qquad \text{is said extremal for } \mathcal{F}.$$

Extremal norms

A further generalization (Elsner (1995)):

$$\rho(\mathcal{F}) \;=\; \inf_{\|\cdot\| \in \mathcal{N}} \|\mathcal{F}\| \quad \text{with } \|\mathcal{F}\| = \max_{A \in \mathcal{F}} \|A\|;$$

where $\mathcal{N}$ is the set of operator norms ($\Longrightarrow \; \rho(\mathcal{F}) \leq \|\mathcal{F}\|$).
When the inf is a min the family is said to be non-defective,

$$\|\cdot\|_\star \;\longrightarrow\; \min_{\|\cdot\| \in \mathcal{N}} \|\mathcal{F}\| \qquad \text{is said extremal for } \mathcal{F}.$$

Example
If $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$, A_i symmetric, the spectral norm is extremal.

Note: for a single matrix the existence of an extremal norm may be deduced by the boundedness of the powers $(A/\rho(A))^n$.

Finiteness property

Definition. Any product $P \in \Sigma_n(\mathcal{F})$ satisfying

$$\rho(\mathcal{F}) = \rho(P)^{1/n}$$

is called a spectrum maximizing product (in short an s.m.p.).

Finiteness property

Definition. Any product $P \in \Sigma_n(\mathcal{F})$ satisfying

$$\rho(\mathcal{F}) = \rho(P)^{1/n}$$

is called a spectrum maximizing product (in short an s.m.p.).
A set $\mathcal{F}$ has the finiteness property if it has at least an s.m.p.

Finiteness property

Definition. Any product $P \in \Sigma_n(\mathcal{F})$ satisfying

$$\rho(\mathcal{F}) = \rho(P)^{1/n}$$

is called a spectrum maximizing product (in short an s.m.p.).
A set $\mathcal{F}$ has the finiteness property if it has at least an s.m.p.

The finiteness conjecture formulated by Lagarias & Wang (1995) asserted that every finite family $\mathcal{F}$ has the finiteness property. The conjecture has been proved to be false.
A simple counterexample is given by $\mathcal{F}_b = \{A, B\}$ with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{with a fixed } b \in (0,1)$$

For uncountably many values of b, the family $\mathcal{F}_b$ has no s.m.p.

Finiteness properties and stability

The interest in finiteness properties is related to its connection with stability of sets of matrices and their many applications.

Finiteness properties and stability

The interest in finiteness properties is related to its connection with stability of sets of matrices and their many applications.

- There is no known algorithm for deciding uniform asymptotic stability of a set of matrices.
- It is known (Blondel & Tsitsiklis, 1997) that there is no algorithm able to approximate (with an a priori accuracy) the joint spectral radius in polynomial time.

Finiteness properties and stability

The interest in finiteness properties is related to its connection with stability of sets of matrices and their many applications.

- There is no known algorithm for deciding uniform asymptotic stability of a set of matrices.
- It is known (Blondel & Tsitsiklis, 1997) that there is no algorithm able to approximate (with an a priori accuracy) the joint spectral radius in polynomial time.
- ○ Stability is algorithmically decidable for sets of matrices that have the finiteness property.

Finiteness properties and stability

The interest in finiteness properties is related to its connection with stability of sets of matrices and their many applications.

- There is no known algorithm for deciding uniform asymptotic stability of a set of matrices.
- It is known (Blondel & Tsitsiklis, 1997) that there is no algorithm able to approximate (with an a priori accuracy) the joint spectral radius in polynomial time.
  - Stability is algorithmically decidable for sets of matrices that have the finiteness property.
  - It has been recently conjectured (Jüngers & Blondel, 2008) that sets of rational matrices have the finiteness property. This would imply that stability is decidable for this important subclass of sets of matrices.

Finiteness properties of rational sets

Theorem (Jüngers & Blondel, 2008). The finiteness property holds for all finite sets of rational matrices $\iff$ it holds for every pair of sign-matrices (i.e. having entries in $\{-1, 0, 1\}$).

Finiteness properties of rational sets

Theorem (Jüngers & Blondel, 2008). The finiteness property holds for all finite sets of rational matrices $\iff$ it holds for every pair of sign-matrices (i.e. having entries in $\{-1, 0, 1\}$).

Conjecture (Jüngers & Blondel, 2008).
Every pair of sign-matrices has the finiteness property.

Finiteness properties of rational sets

Theorem (Jüngers & Blondel, 2008). The finiteness property holds for all finite sets of rational matrices $\iff$ it holds for every pair of sign-matrices (i.e. having entries in $\{-1, 0, 1\}$).

Conjecture (Jüngers & Blondel, 2008).
Every pair of sign-matrices has the finiteness property.

Theorem (Cicone, G. & Serra Capizzano, 2008).
Every pair of 2×2 sign-matrices has the finiteness property.

This result could be used as a basis for an induction argument which however appears non-trivial.

Finiteness properties of rational sets

Theorem (Jüngers & Blondel, 2008). The finiteness property holds for all finite sets of rational matrices $\iff$ it holds for every pair of sign-matrices (i.e. having entries in $\{-1, 0, 1\}$).

Conjecture (Jüngers & Blondel, 2008).
Every pair of sign-matrices has the finiteness property.

Theorem (Cicone, G. & Serra Capizzano, 2008).
Every pair of 2×2 sign-matrices has the finiteness property.

This result could be used as a basis for an induction argument which however appears non-trivial.

Conjecture (Cicone, G. & Serra Capizzano, 2008).
Let $\mathcal{F}$ be a pair of $n \times n$ sign-matrices. The maximal length ℓ_n of a minimal s.m.p. (that is an s.m.p. which is not a power of another s.m.p.) fulfils the inequality $\ell_n \leq n^3$.

Theoretical results

A useful scaling: Let $Q \in \Sigma_n(\mathcal{F})$ a certain product of lenth n with $\rho(Q) \neq 0$; we divide $\mathcal{F}$ by the scalar $\vartheta := \rho(Q)^{1/n}$, i.e.

$$\mathcal{F}^* = \{A_i^*\}_{i \in \mathcal{I}} \quad \text{with } A_i^* = \frac{1}{\vartheta} A_i \qquad \text{(scaled family)}.$$

Theoretical results

A useful scaling: Let $Q \in \Sigma_n(\mathcal{F})$ a certain product of lenth n with $\rho(Q) \neq 0$; we divide $\mathcal{F}$ by the scalar $\vartheta := \rho(Q)^{1/n}$, i.e.

$$\mathcal{F}^* = \{A_i^*\}_{i \in \mathcal{I}} \quad \text{with } A_i^* = \frac{1}{\vartheta} A_i \qquad \text{(scaled family)}.$$

Basic property. The scaled family $\mathcal{F}^*$ is such that $\rho(\mathcal{F}^*) \geq 1$.

Theoretical results

A useful scaling: Let $Q \in \Sigma_n(\mathcal{F})$ a certain product of lenth n with $\rho(Q) \neq 0$; we divide $\mathcal{F}$ by the scalar $\vartheta := \rho(Q)^{1/n}$, i.e.

$$\mathcal{F}^* = \{A_i^*\}_{i\in\mathcal{I}} \quad \text{with } A_i^* = \frac{1}{\vartheta} A_i \qquad \text{(scaled family)}.$$

Basic property. The scaled family $\mathcal{F}^*$ is such that $\rho(\mathcal{F}^*) \geq 1$.

Consider the multiplicative semigroup

$$\Sigma(\mathcal{F}^*) = \bigcup_{n\geq 1} \Sigma_n(\mathcal{F}^*) = \bigcup_{n\geq 1} \bigcup_{i_1,\ldots,i_n\in\mathcal{I}} A_{i_n}\cdots A_{i_1}$$

Theoretical results

A useful scaling: Let $Q \in \Sigma_n(\mathcal{F})$ a certain product of lenth n with $\rho(Q) \neq 0$; we divide $\mathcal{F}$ by the scalar $\vartheta := \rho(Q)^{1/n}$, i.e.

$$\mathcal{F}^* = \{A_i^*\}_{i \in \mathcal{I}} \quad \text{with } A_i^* = \frac{1}{\vartheta} A_i \qquad \text{(scaled family)}.$$

Basic property. The scaled family $\mathcal{F}^*$ is such that $\rho(\mathcal{F}^*) \geq 1$.

Consider the multiplicative semigroup

$$\Sigma(\mathcal{F}^*) = \bigcup_{n \geq 1} \Sigma_n(\mathcal{F}^*) = \bigcup_{n \geq 1} \bigcup_{i_1,\ldots,i_n \in \mathcal{I}} A_{i_n} \cdots A_{i_1}$$

Result 1 (Barabanov, 1988): if $\Sigma(\mathcal{F}^*)$ is bounded then $\mathcal{F}^*$ has an extremal norm and $\rho(\mathcal{F}^*) = 1$. This implies that $\rho(\mathcal{F}) = \vartheta$, Q is an s.m.p. and $\mathcal{F}$ has the finiteness property.

Construction of an extremal norm

Result 2: if $\Sigma(\mathcal{F}^*)$ is bounded and $x \in \mathbf{R}^k$ is such that the set $\mathcal{T}(x) = \Sigma(\mathcal{F}^*)x$ spans $\mathbf{R}^k$ then the convex hull

$$\mathcal{P} := \mathrm{co}(\mathcal{T}(x), \mathcal{T}(-x)) \qquad \text{(which is symmetric)}$$

is an invariant set for $\mathcal{F}^*$ and unit ball of an extremal norm.

Construction of an extremal norm

Result 2: if $\Sigma(\mathcal{F}^*)$ is bounded and $x \in \mathbf{R}^k$ is such that the set $\mathcal{T}(x) = \Sigma(\mathcal{F}^*)x$ spans $\mathbf{R}^k$ then the convex hull

$$\mathcal{P} := \mathrm{co}(\mathcal{T}(x), \mathcal{T}(-x)) \qquad \text{(which is symmetric)}$$

is an invariant set for $\mathcal{F}^*$ and unit ball of an extremal norm.

Remark: the set $\mathcal{T}(x)$ is not finite in general.

Construction of an extremal norm

Result 2: if $\Sigma(\mathcal{F}^*)$ is bounded and $x \in \mathbf{R}^k$ is such that the set $\mathcal{T}(x) = \Sigma(\mathcal{F}^*)x$ spans $\mathbf{R}^k$ then the convex hull

$$\mathcal{P} := \mathrm{co}(\mathcal{T}(x), \mathcal{T}(-x)) \qquad \text{(which is symmetric)}$$

is an invariant set for $\mathcal{F}^*$ and unit ball of an extremal norm.

Remark: the set $\mathcal{T}(x)$ is not finite in general.

Result 3 (G., Wirth & Zennaro, 2005): if Q is an s.m.p. and x is a leading eigenvector of Q (+ some technical assumption) the set $\mathcal{P}$ is finitely generated (and hence is a polytope), i.e.

$$\mathcal{P} = \mathrm{co}\left(\pm P_1^* x, P_2^* x, \ldots, \pm P_s^* x\right),$$

with $P_1^*, P_2^*, \ldots, P_s^*$ certain finite products in $\Sigma(\mathcal{F}^*)$.

How to get extremality results.

Polytope extremal norms imply s.m.p. (Lagarias–Wang, 1995).

Do s.m.p. imply polytope extremal norms? Unfortunately not always (counterexamples have been found recently). However this holds adding some assumptions (G. & Zennaro, 2008).

How to get extremality results.

Polytope extremal norms imply s.m.p. (Lagarias–Wang, 1995).

Do s.m.p. imply polytope extremal norms? Unfortunately not always (counterexamples have been found recently). However this holds adding some assumptions (G. & Zennaro, 2008).

Basic tool: • Look for a candidate spectrum maximizing product $Q \in \Sigma_n(\mathcal{F})$ and scale the set of matrices by $\vartheta = \rho(Q)^{1/n}$ in order to get a scaled set $\mathcal{F}^*$ with $\rho(\mathcal{F}^*) \geq 1$.

How to get extremality results.

Polytope extremal norms imply s.m.p. (Lagarias–Wang, 1995).

Do s.m.p. imply polytope extremal norms? Unfortunately not always (counterexamples have been found recently). However this holds adding some assumptions (G. & Zennaro, 2008).

Basic tool: • Look for a candidate spectrum maximizing product $Q \in \Sigma_n(\mathcal{F})$ and scale the set of matrices by $\vartheta = \rho(Q)^{1/n}$ in order to get a scaled set $\mathcal{F}^*$ with $\rho(\mathcal{F}^*) \geq 1$.

• Then look for an invariant convex symmetric set for $\mathcal{F}^*$. If the procedure succeeds then $\rho(\mathcal{F}^*) = 1$, that means that the j.s.r. of $\mathcal{F}$ is also computed and $\mathcal{F}$ has the finiteness property.

How to get extremality results.

Polytope extremal norms imply s.m.p. (Lagarias–Wang, 1995).

Do s.m.p. imply polytope extremal norms? Unfortunately not always (counterexamples have been found recently). However this holds adding some assumptions (G. & Zennaro, 2008).

Basic tool: • Look for a candidate spectrum maximizing product $Q \in \Sigma_n(\mathcal{F})$ and scale the set of matrices by $\vartheta = \rho(Q)^{1/n}$ in order to get a scaled set $\mathcal{F}^*$ with $\rho(\mathcal{F}^*) \geq 1$.

• Then look for an invariant convex symmetric set for $\mathcal{F}^*$. If the procedure succeeds then $\rho(\mathcal{F}^*) = 1$, that means that the j.s.r. of $\mathcal{F}$ is also computed and $\mathcal{F}$ has the finiteness property.

• By an algorithm which computes recursively the set $\Sigma(\mathcal{F}^*)x$ (for a suitable initial vector) one has that if the algorithm halts then the resulting invariant set gives a polytope extremal norm.

Setting an algorithm

Let $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ be a finite family; choose a candidate s.m.p. $Q \in \Sigma_n(\mathcal{F})$. Let x be the leading eigenvector of Q.

Setting an algorithm

Let $\mathcal{F} = \{A_i\}_{i\in\mathcal{I}}$ be a finite family; choose a candidate s.m.p. $Q \in \Sigma_n(\mathcal{F})$. Let x be the leading eigenvector of Q.

Set $\vartheta = \rho(Q)^{1/n}$ and define the scaled family

$$\mathcal{F}^* = \{\vartheta^{-1} A_i\}_{i\in\mathcal{I}} \quad \text{with } \rho(\mathcal{F}^*) \geq 1.$$

Setting an algorithm

Let $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ be a finite family; choose a candidate s.m.p. $Q \in \Sigma_n(\mathcal{F})$. Let x be the leading eigenvector of Q.

Set $\vartheta = \rho(Q)^{1/n}$ and define the scaled family

$$\mathcal{F}^* = \{\vartheta^{-1} A_i\}_{i \in \mathcal{I}} \quad \text{with } \rho(\mathcal{F}^*) \geq 1.$$

Compute recursively the set $\mathcal{T}(x)$, that is initialize $\mathcal{T}^{(0)}(x) = x$ and compute

$$\mathcal{T}^{(m+1)}(x) = \mathcal{F}^* \mathcal{T}^{(m)}(x), \qquad m \geq 0.$$

Setting an algorithm

Let $\mathcal{F} = \{A_i\}_{i \in \mathcal{I}}$ be a finite family; choose a candidate s.m.p. $Q \in \Sigma_n(\mathcal{F})$. Let x be the leading eigenvector of Q.

Set $\vartheta = \rho(Q)^{1/n}$ and define the scaled family

$$\mathcal{F}^* = \{\vartheta^{-1} A_i\}_{i \in \mathcal{I}} \quad \text{with } \rho(\mathcal{F}^*) \geq 1.$$

Compute recursively the set $\mathcal{T}(x)$, that is initialize $\mathcal{T}^{(0)}(x) = x$ and compute

$$\mathcal{T}^{(m+1)}(x) = \mathcal{F}^* \mathcal{T}^{(m)}(x), \qquad m \geq 0.$$

Check whether $\mathrm{co}\,(\mathcal{T}(x), \mathcal{T}(-x))$ is a polytope that is - at any step - if $\mathrm{co}\left(\mathcal{T}^{(m)}(x), \mathcal{T}^{(m)}(-x)\right)$ is an invariant set for $\mathcal{F}^*$.

Application to a pair of sign-matrices

Consider the family $\mathcal{F} = \{A, B\}$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

having s.m.p.: $P = A\,B\,A^2\,B$.

We consider two situations.

Application to a pair of sign-matrices

Consider the family $\mathcal{F} = \{A, B\}$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

having s.m.p.: $P = A\,B\,A^2\,B$.

We consider two situations.

In the first one we consider as a candidate s.m.p. $Q_1 = B^2$,

which is a wrong guess.

Application to a pair of sign-matrices

Consider the family $\mathcal{F} = \{A, B\}$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

having s.m.p.: $P = A\,B\,A^2\,B$.

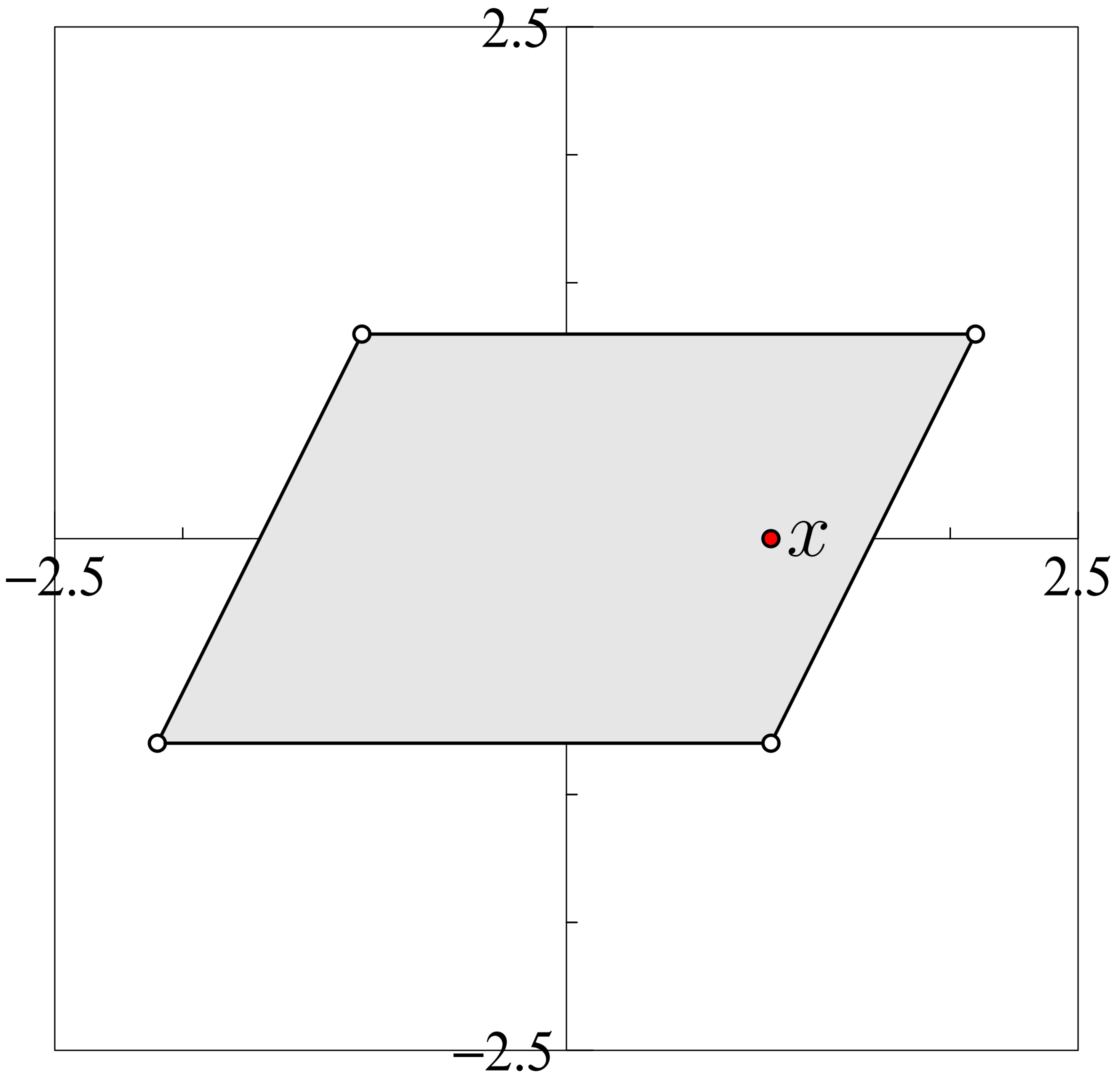
We consider two situations.

In the first one we consider as a candidate s.m.p. $Q_1 = B^2$, which is a wrong guess.

In the second case we consider as a candidate s.m.p. $Q_2 = P$, that is a right guess.

Case 1

Let $\vartheta = \sqrt[2]{\rho(Q_1)}$ and set $\mathcal{F}^* = \{A^*, B^*\} = \{A/\vartheta, B/\vartheta\}$.



$\mathrm{co}\left(\mathcal{T}^{(3)}(x), \mathcal{T}^{(3)}(-x)\right)$

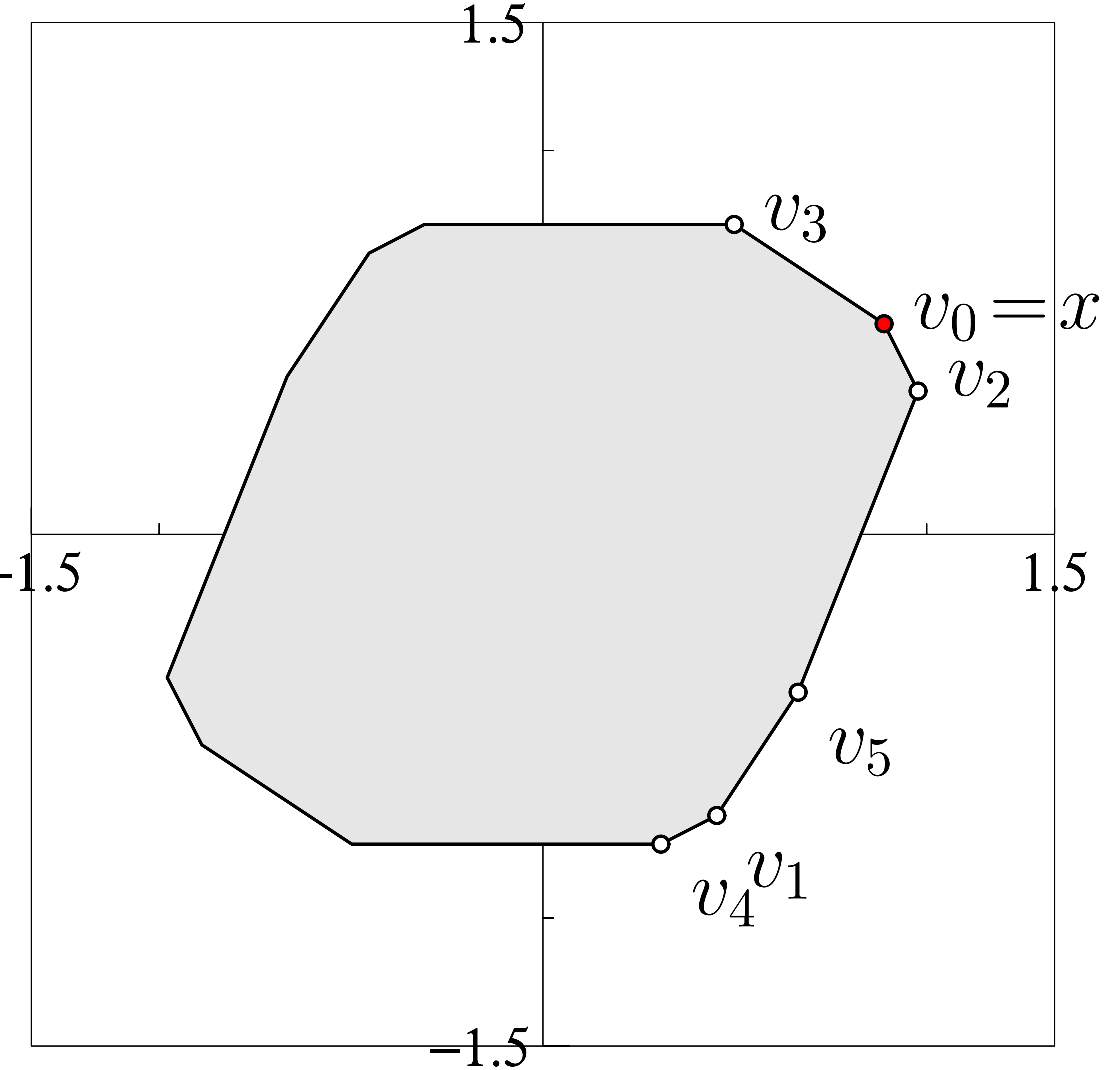
Theorem.

If x is internal to the set $\mathrm{co}\left(\mathcal{T}^{(m)}(x), \mathcal{T}^{(m)}(-x)\right)$ for some m then

$$\rho(\mathcal{F}^*) > 1.$$

Case 2

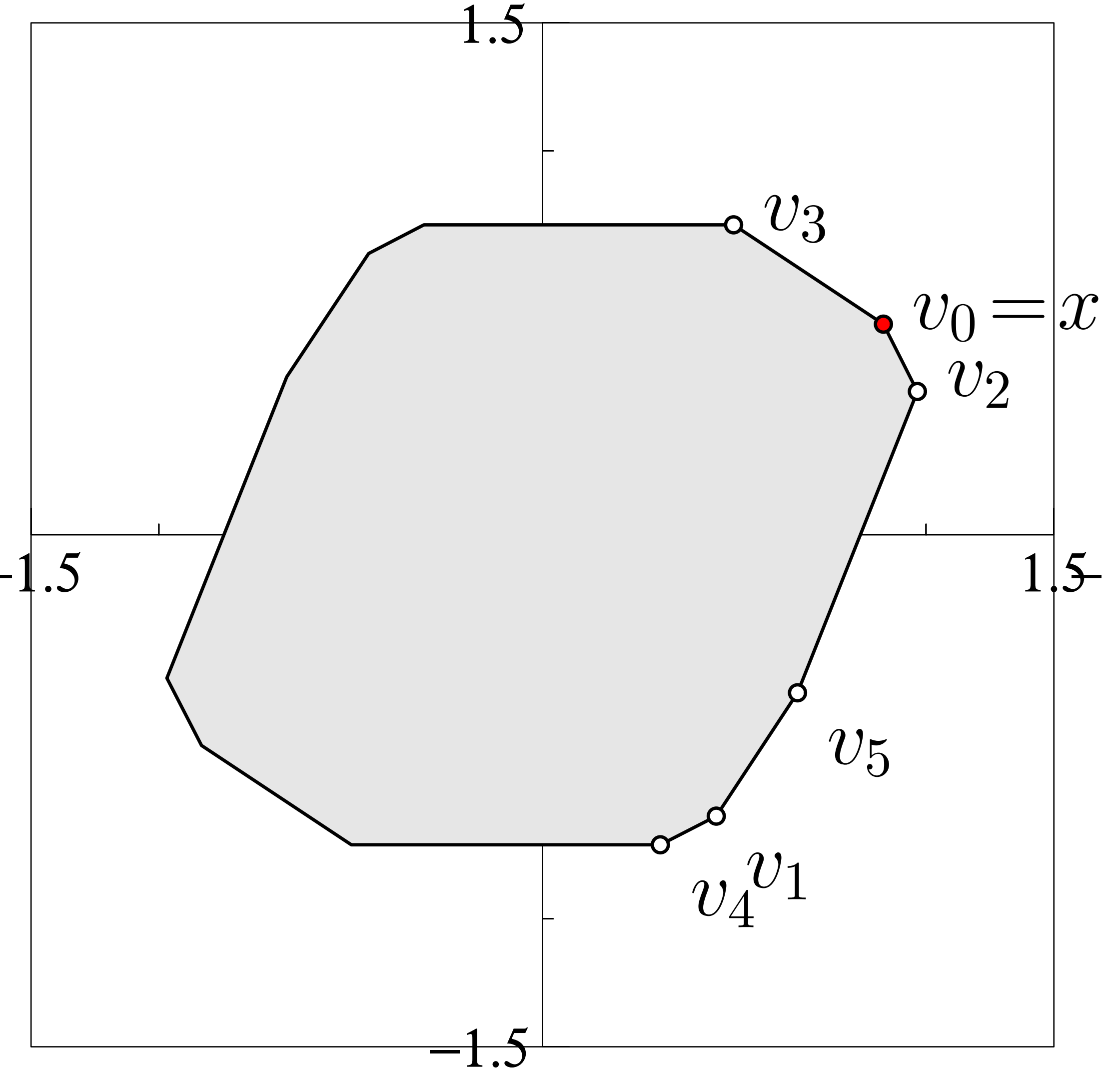
Let $\vartheta = \sqrt[5]{\rho(P)}$ and set $\mathcal{F}^* = \{A^*, B^*\} = \{A/\vartheta, B/\vartheta\}$.

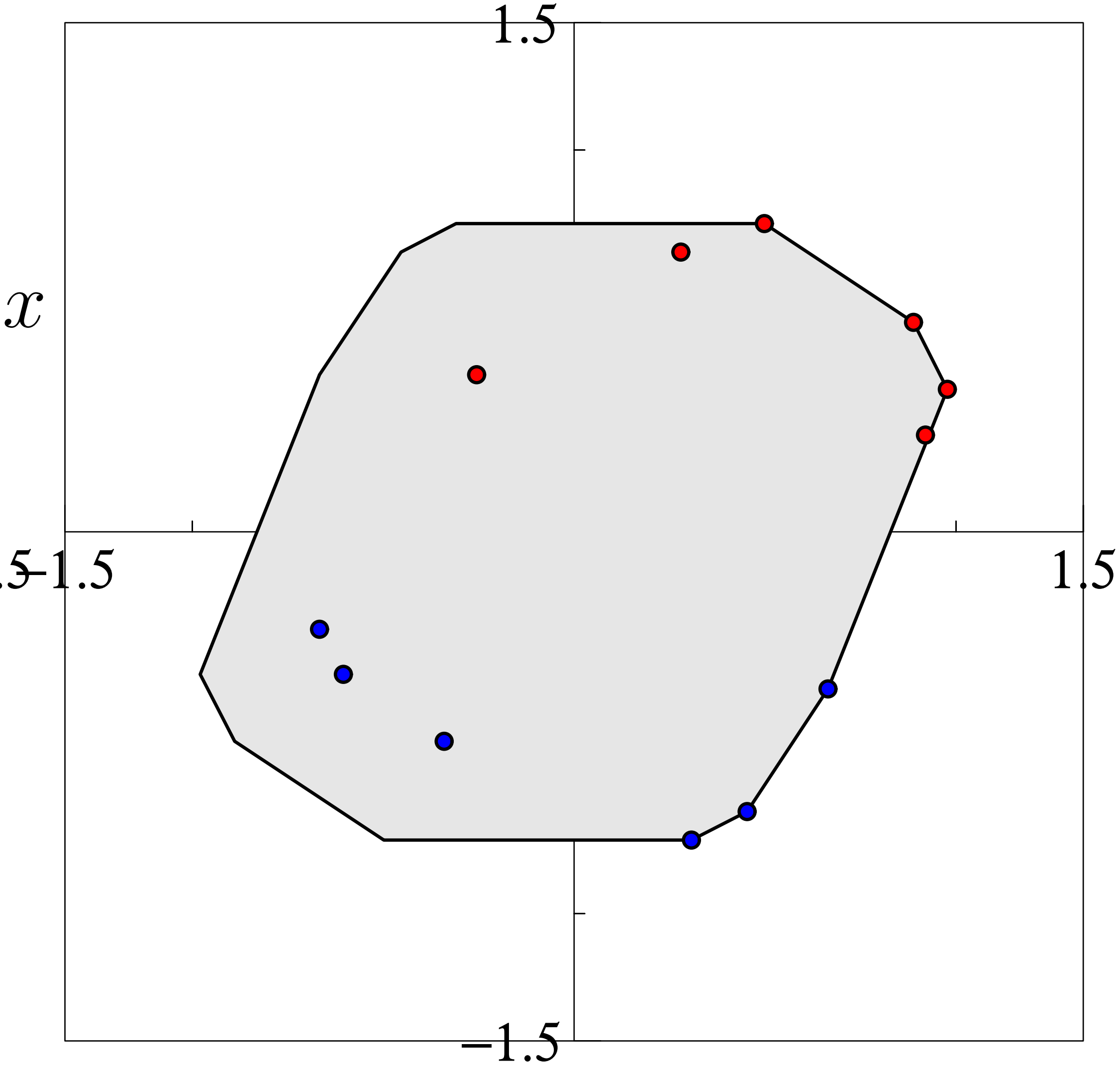


The extremal polytope norm $\mathcal{P} = \mathrm{co}\left(T^{(4)}(x), T^{(4)}(-x)\right)$.

Case 2

Let $\vartheta = \sqrt[5]{\rho(P)}$ and set $\mathcal{F}^* = \{A^*, B^*\} = \{A/\vartheta, B/\vartheta\}$.





The extremal polytope norm $\mathcal{P} = \mathrm{co}\left(T^{(4)}(x), T^{(4)}(-x)\right)$.

Final remarks

The property that an s.m.p. has a real leading eigenvector is not generic.

Final remarks

The property that an s.m.p. has a real leading eigenvector is not generic. In the other generic case there are two complex conjugate eigenvectors.

Final remarks

The property that an s.m.p. has a real leading eigenvector is not generic. In the other generic case there are two complex conjugate eigenvectors. For example $\mathcal{F} = \{A, B\}$ ($\rho(\mathcal{F}) = 1$)

$$A = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Final remarks

The property that an s.m.p. has a real leading eigenvector is not generic. In the other generic case there are two complex conjugate eigenvectors. For example $\mathcal{F} = \{A, B\}$ ($\rho(\mathcal{F}) = 1$)

$$A = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Starting from the leading eigenvector of B, construct $\Sigma(\mathcal{F})x$:

$$x = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}; \qquad v_n := A^n x = \begin{pmatrix} \cos(n) & \sin(n) \end{pmatrix}^{\mathrm{T}}.$$

This is dense on the unit disk and gives (asymptotically) the 2-norm as extremal.

Final remarks

The property that an s.m.p. has a real leading eigenvector is not generic. In the other generic case there are two complex conjugate eigenvectors. For example $\mathcal{F} = \{A, B\}$ $(\rho(\mathcal{F}) = 1)$

$$A = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Starting from the leading eigenvector of B, construct $\Sigma(\mathcal{F})x$:

$$x = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}; \qquad v_n := A^n x = \begin{pmatrix} \cos(n) & \sin(n) \end{pmatrix}^{\mathrm{T}}.$$

This is dense on the unit disk and gives (asymptotically) the 2-norm as extremal. However $\exists$ a complex polytope norm...