Structured Linear Algebra Problems:
Analysis, Algorithms, and Applications

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Structured matrices in nonlinear imaging

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Outline

I - Nonlinear Inverse Problems in imaging (linear vs nonlinear case).

II - The block matrix of the linearization iterative scheme.

III - Exploiting the structure of the blocks:
direct and iterative regularization blocks methods.

IV - A three-level block splitting iterative regularization algorithm.

V - SuperResolution post-processing enhancement.

VI - Numerical results.

Inverse Problem

By the knowledge of some "observed" data y (i.e., the effect),
find an approximation of some model parameters x (i.e., the cause).

Usually, inverse problems are ill-posed, they need regularization technique.

NonLinear Inverse Problem

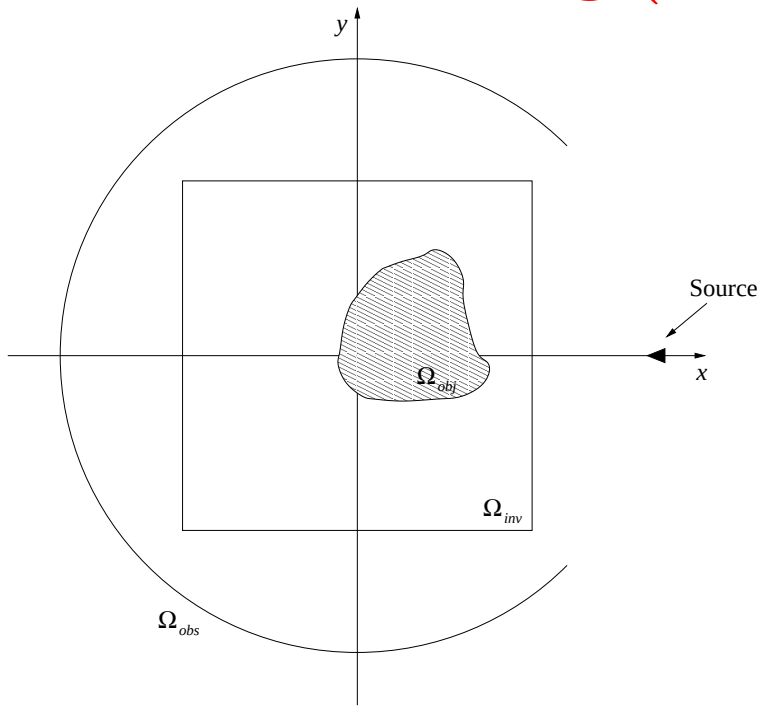
Given the data $y \in Y$, find (an approximation of) the unknown $x \in X$ such that

$$A(x) = y$$

where $A : X \longrightarrow Y$ is a **nonlinear** operator (Fréchet differentiable), between the Hilbert spaces X and Y.

A nonlinear inverse problem:

The Microwave Inverse Scattering (nonlinear imaging)



Input: scattered electromagnetic field on Ω_{obs} (observation domain).
Output: dielectric properties, i.e. the object, in Ω_{inv} (investigation domain).
The model leads to a nonlinear integral equation -particles' interaction-.

Features: very low degree of invasivity; provide information about the dielectric properties (instead of density); microwave cheap and easy to generate.
Applications: medical imaging, nondestructive evaluations of materials, subsurface prospecting,...

Linear Imaging vs Nonlinear Imaging

Inverse problem in Imaging: to reconstruct the true image x from the knowledge of the acquired image $A(x)$, where A is an integral operator.

Linear imaging (Image Deblurring)

$$(A\,x)(r) = \int_{\Omega_{inv}} G(r - r')\, x(r')\, dr'$$

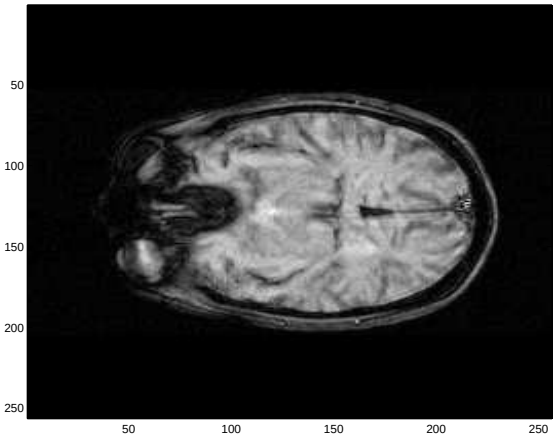
$\forall r \in \Omega_{inv}$, i.e. the 2D investigation domain.

Nonlinear imaging (Inverse Scattering)

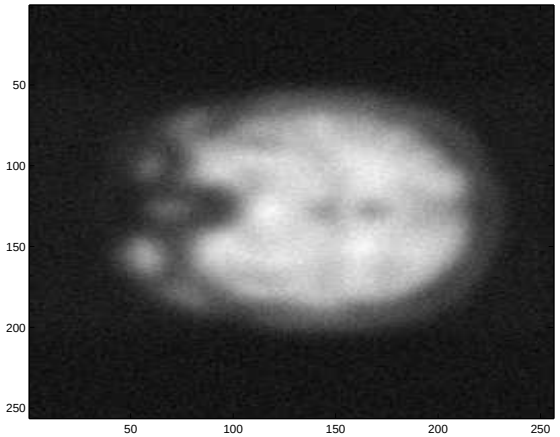
$$(A(x))(r) = \int_{\Omega_{inv}} G(r - r')\, (N(x))(r')\, dr'$$

$\forall r \in \Omega_{obs}$, i.e. the 2D observation domain, where N is a nonlinear functional (sometimes not completely known).

Linear Imaging

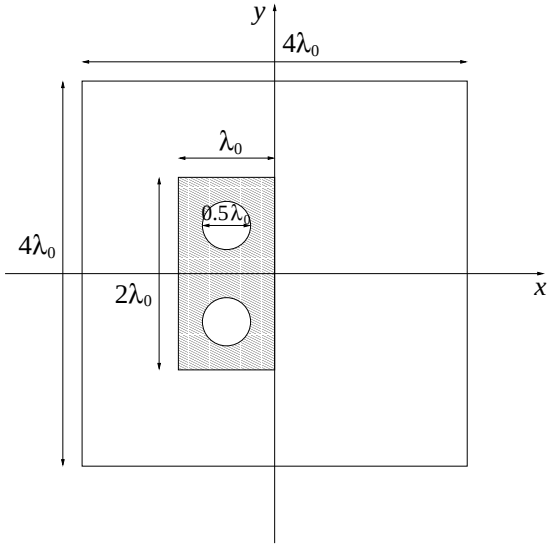


True object to restore

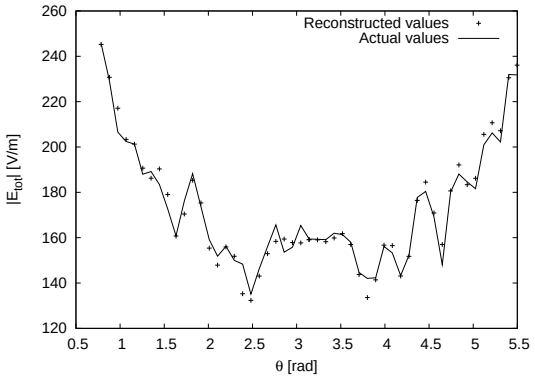


Input data for restoration

Nonlinear imaging



True object to restore



Input data for restoration

Microwave Inverse Scattering

Forward non-linear integral operator $A : D \longrightarrow E$, where $D = L^2(\mathbb{R}^2)$ the Dielectric permittivity, and $E = L^2(\mathbb{R}^2)$ the scattered Electric field,

$$(A(\chi))(r) = \int_{\Omega_{inv}} G(r - r') \left[(A(\chi))(r') + u_{inc}(r') \right] \chi(r')\, dr'$$

$$\forall r \in \Omega_{obs}.$$

$\chi \in D$ is the scatterer (i.e., the unknown to retrieve),
$A(\chi)$ is the scattered field, unknown in Ω_{inv},
u_{inc} is the incident field, known everywhere, G is the known integral kernel.

Inverse Scattering Problem

INPUT: the scattered field $u_{scat} = A(\chi)$ on the observation domain Ω_{obs}
OUTPUT: the scattering potential χ in the investigation domain Ω_{inv}

by solving of the nonlinear equation $\quad A(\chi) = u_{scat}$

Coupled formulation for both scatterer and scattered field

Since $u_{tot} = u_{scat} + u_{inc}$ is not known inside the investigation domain Ω_{inv}, we have to consider it as unknown too (together with the actual unknown χ to retrieve). We obtain two coupled integral equations.

In the observation domain Ω_{obs} (i.e., measured data):

$$\int_{\Omega_{inv}} G(r - r')\, u_{tot}(r')\, \chi(r')\, dr' = u_{scat}(r) \qquad \forall r \in \Omega_{obs}.$$

In the investigation domain Ω_{inv}:

$$u_{tot}(r) - \int_{\Omega_{inv}} G(r - r')\, u_{tot}(r')\, \chi(r')\, dr' = u_{inc}(r) \qquad \forall r \in \Omega_{inv}.$$

Remarks The apparatus is rotated (multiple views), in order to provide different acquisitions of u_{scat} onto Ω_{obs}. In the model we have to consider an index $p = 1, \ldots, P$ related to the particular view.

Full Formulation for both scatterer and scattered field

By introducing the nonlinear operator A defined as

$$A(u^1_{tot}, \dots, u^P_{tot}, \chi)(r) = \begin{pmatrix} \int_{\Omega_{inv}} G(r-r')\, u^1_{tot}(r')\, \chi(r')\, dr' \\ \vdots \\ \int_{\Omega_{inv}} G(r-r')\, u^P_{tot}(r')\, \chi(r')\, dr' \\ u^1_{tot}(r) - \int_{\Omega_{inv}} G(r-r')\, u^1_{tot}(r')\, \chi(r')\, dr' \\ \vdots \\ u^P_{tot}(r) - \int_{\Omega_{inv}} G(r-r')\, u^P_{tot}(r')\, \chi(r')\, dr' \end{pmatrix}$$

and the data vector $\quad b = \left(u^1_{scat}, \dots, u^P_{scat}, u^1_{inc}, \dots, u^P_{inc}\right)^T,$

the **inverse scattering problem** becomes:

find $\chi \in L^2(\Omega_{inv})$ and $u^s_{tot} \in L^2(\Omega_{inv})$, $s = 1, \dots, P$, such that

$$A(u^1_{tot}, \dots, u^P_{tot}, \chi) = b$$

The computation of the Fréchet Derivative for linearization

The Fréchet Derivative of the operator A at the point x is the linear operator $A'_x : X \longrightarrow Y$ such that $A(x+h) = A(x) + A'_x h + O(\|h\|^2)$.

The linearization gives rise to the sparse and structured matrix:

$$A'_x = \begin{pmatrix} A_{\chi,1} & 0 & \dots & 0 & A_{u,1} \\ 0 & A_{\chi,2} & \ddots & \vdots & A_{u,2} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \ddots & \ddots & A_{\chi,P} & A_{u,P} \\ I - A_\chi & \ddots & \ddots & 0 & -A_{u,1} \\ 0 & I - A_\chi & \ddots & \vdots & -A_{u,2} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & I - A_\chi & -A_{u,P} \end{pmatrix}$$

where the **linear** operators $\{A_{u,j}\}_{j=1,\dots,P}$, $\{A_{\chi,j}\}_{j=1,\dots,P}$, and A_χ are

$$A_{\chi,j} h(r) = \int_{\Omega_{inv}} G(r-r')\,\chi(r')\,h(r')\,dr' \qquad r \in \Omega^j_{obs}$$

$$A_{u,j} h(r) = \int_{\Omega_{inv}} G(r-r')\,h(r')\,u^j_{tot}(r')\,dr' \qquad r \in \Omega^j_{obs}$$

$$A_\chi h(r) = \int_{\Omega_{inv}} G(r-r')\,\chi(r')\,h(r')\,dr' \qquad r \in \Omega_{inv}$$

In such problems, the dimensions of the matrices are extremely large in general. Here we have:

- $n \times n$ discretization of Ω_{inv}
- m receivers on Ω_{obs}
- $P = S \cdot F$, where S is the number of views (rotations of the apparatus) and F the number of illuminations (different frequencies of the incident wave)

The **size** of the matrix A'_x is

$$(P(m + n^2)) \times ((P + 1)n^2)$$

Real example: Data from Institut Fresnel, Marseille:
$m = 241,\ S = 18,\ F = 9$ (diameter of Ω_{obs} is about 3 m., $P = S \cdot F = 164$).

With a (small) $n \times n = 64 \times 64$ discretization, A'_x is $7.0 \cdot 10^5 \times 6.7 \cdot 10^5$
With $n \times n = 1024 \times 1024$, A'_x is about $2 \cdot 10^8 \times 2 \cdot 10^8$.

Exploiting the structure of the blocks (I)

Any block matrix of A'_x is a linear operators like

$$(S\,h)(r) = \int_{\Omega_{inv}} s(r, r')\, h(r')\, dr' \,,$$

where the integral kernel s is always known and is given by the product of the shift invariant kernel G times a fixed function depending on x.

This kind of operators is related to Toeplitz-like X diagonal matrices.

In particular, $S\,h$ needs to be evaluated for

- either the investigation domain $r \in \Omega_{inv}$
- or the observation domain $r \in \Omega_{obs}$.

Since Ω_{inv} and Ω_{obs} are different, then A'_x is a submatrix (i.e., a Low Rank Extracted System) of a block matrix with Toeplitz-times-diagonal blocks.

Exploiting the structure of the blocks (II)

How can we efficiently compute any matrix-vector product Sh?

To use FFT-based matrix product, the matrix A'_x must be embedded in a full block matrix with Toeplitz-times-diagonal blocks (related to a rectangular discretization domain containing both Ω_{inv} and Ω_{obs}).

Does the FFT-based matrix product really reduce the numerical complexity in real applications?

For the blocks related to Ω_{obs} the answer is negative, since usually Ω_{inv} is far from Ω_{obs}.

For the blocks related to Ω_{inv} the answer is positive. In particular, here the use of the anti-reflective matrix algebra can be useful for some application such as subsurface prospecting (see the next talk...).

Dealing with the least square equation

Any linearization step, the least square equation to regularize gives rise to the following arrow block-matrix

$$E = A_x'^{*} A_x' = \begin{pmatrix} M_1 & & & & V_1 \\ & M_2 & & & V_2 \\ & & \ddots & & \vdots \\ & & & M_P & V_P \\ V_1^* & V_2^* & \dots & V_P^* & C \end{pmatrix}$$

where any block is the sum of products of structured matrices.

Solving schemes for any linearized step:

I **Regularized Direct Blocks methods**

Block Cholesky Factorization with Tikhonov regularization

II **Regularizing Iterative Blocks methods**

Block Decomposition with regularizing iterative solution of any block system

Regularized Direct Blocks methods

Tikhonov regularized linear system $(A_x'^{\,*} A_x' + \mu I)h = A_x'^{\,*} r$.

The Block Cholesky factorization of the block arrow matrix $\tilde{A}_x'^{\,*} A_x' + \mu I$ does not produce fill-in. Thus, we have

$$\tilde{A}_x'^{\,*} A_x' + \mu I = L L^* \,,$$

$$L = \begin{pmatrix} L_1 & & & & \\ & L_2 & & & \\ & & \ddots & & \\ & & & L_P & \\ \hat{L}_1 & \hat{L}_2 & \dots & \hat{L}_P & \hat{L}_0 \end{pmatrix}, \quad \text{where}$$

- L_p is the Cholesky factor of $M_p + \mu I = L_p L_p^*$, for $p = 1, \dots, P$;
- $\hat{L}_p$ is the full matrix $\hat{L}_p = V_p L_p^{-*}$, for $p = 1, \dots, P$;
- $\hat{L}_0$ is the Cholesky factor of $C - \sum_{p=1}^{P} \hat{L}_p \hat{L}_p^* + \mu I$.

Overall cost: $(8P + 1)n^3/6 + O(Pn^2)$.

Regularizing Iterative Blocks methods

Block splitting of the linear system ${A'_x}^* A'_x h = {A'_x}^* r$

$$A = M - N \quad \text{on the block level}$$

$$M h^{(t+1)} = b - N h^{(t)} \qquad \text{for } t = 0, 1, 2, \ldots$$

Block Jacobi $\qquad M = \text{BlockDiag}({A'_x}^* A'_x)$

Block Gauss-Seidel $\qquad M = \text{BlockTril}({A'_x}^* A'_x)$

Block-arrow matrices are 2-cyclic and consistently ordered, then Gauss-Seidel is doubly faster than Jacobi, since $\varrho(B_G) = (\varrho(B_J))^2$.

Any iteration of the block splitting method requires the solution of $P+1$ small structured system blocks of $n^2 \times n^2$ elements. To solve these inner systems, we adopt iterative regularization methods.

Three nested-levels iterative regularization algorithm for structured block matrices

I- **Outer Iterations: Gauss-Newton Method**

Let $j = 0$ and x_0 be the initial guess. Compute the derivative A'_{x_j}, and find a regularized solution of the linear system

$$A'^{*}_{x_j} A'_{x_j} h_j = A'^{*}_{x_j} \left(b - A(x_j)\right)$$

by means of the nested iterations II-III.

Then update $x_{j+1} = x_j + h_j$, until a stopping rule (discrepancy principle, GCV, ...) holds true.

II- **Inner Iterations: Block level**

Compute a <u>block splitting</u> of the arrow matrix $A'^{*}_{x} A'_{x}$ and solve the related steps by means of the nested iterations III.

III- **Inner Iterations: Nested level**

Compute a <u>regularized solution</u> of each <u>inner block system</u>, involving (each time only few) M_j, V_j, C, for $j = 1, \ldots, P$, with fast techniques of structured numerical linear algebra.

Improving the quality of the output image: SuperResolution postprocessing techniques

Nowadays dense discretizations are still computationally very expensive.

SuperResolution: from a set of Low-Resolution (LR) images to a (better) High-Resolution (HR) image.

SuperResolution is the following linear inverse problem:
given a set of M Low-Resolution images χ_i (i.e., the input), find the High-Resolution image χ (i.e., the output) such that

$$\chi_i = DS_i\chi + \eta_i \qquad\qquad i = 1, \ldots, M$$

- D is the decimation matrix that transform a $ln \times ln$ into a $n \times n$ image,
- S_i is the i-th geometric distortion (i.e., shift and rotation) of χ_i with respect to a reference image,
- η_i is the noise on the i-th LR image.

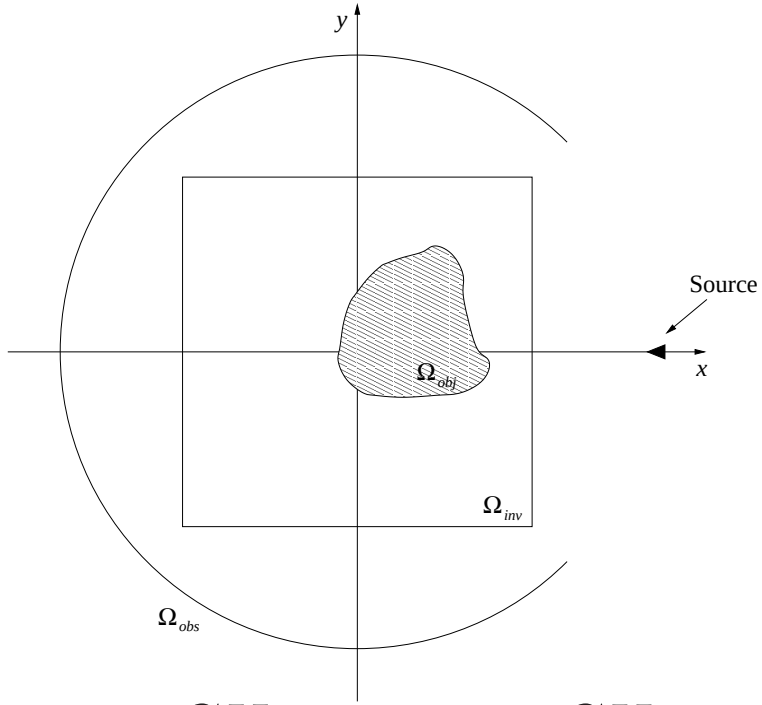
The reconstruction problem for super-resolution amounts to computing χ from the inverse problem

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_r \end{pmatrix} = \begin{pmatrix} DS_1 \\ DS_2 \\ \vdots \\ DS_r \end{pmatrix} \chi + \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_r \end{pmatrix} .$$

The problem is ill-posed, and a regularization procedure for the least square solution is required.

In our implementation, we use the Landweber method with projection on positives.

Numerical results



Frequency: $F = 15$ $(0.6 \div 2$ GHz, step 0.1 GHz, wavelength $50 \div 15$ cm).

Views: $V = 8$ (the apparatus is 8 times rotated by $2\pi/8$).

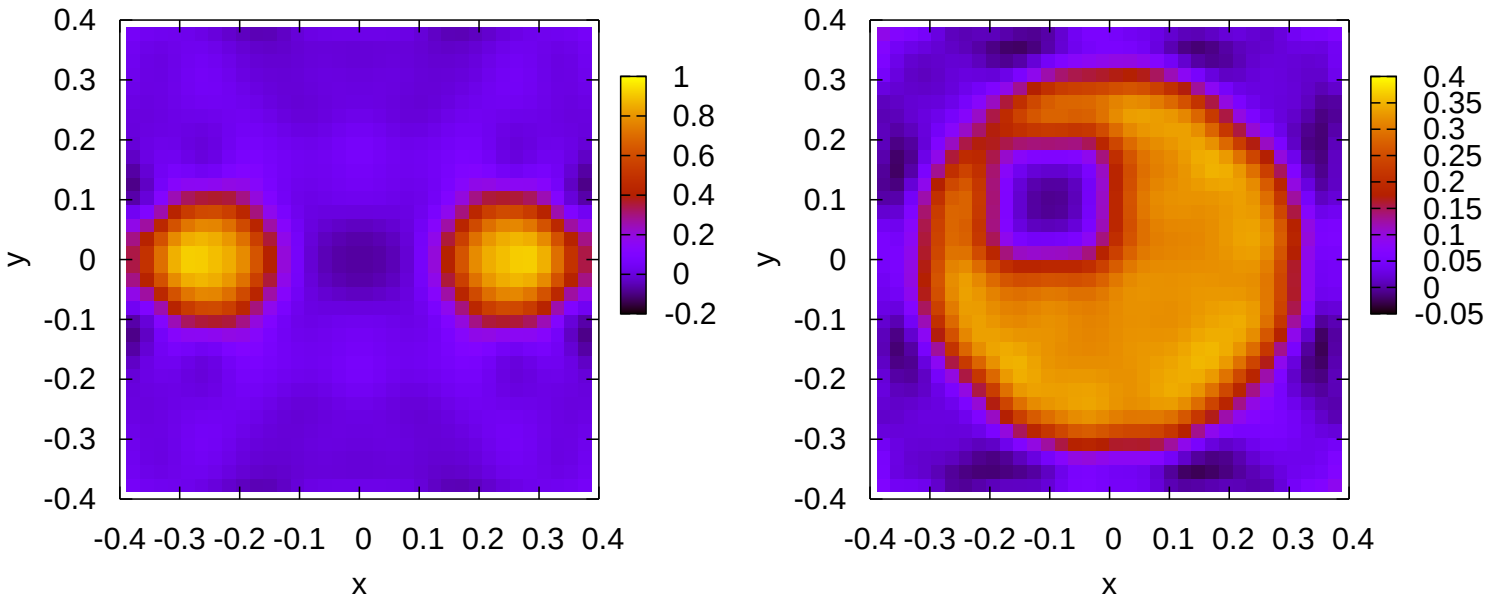
Measurement points: $M = 241$ points equispaced on Ω_{inv}, radius 1.67 m.

Investigation domain: square with side of 1 m., discretization of 31×31 cells.

Noise: Gaussian with 0 mean, SNR=20dB, Relative noise=10%.

Initial guess: empty scene.

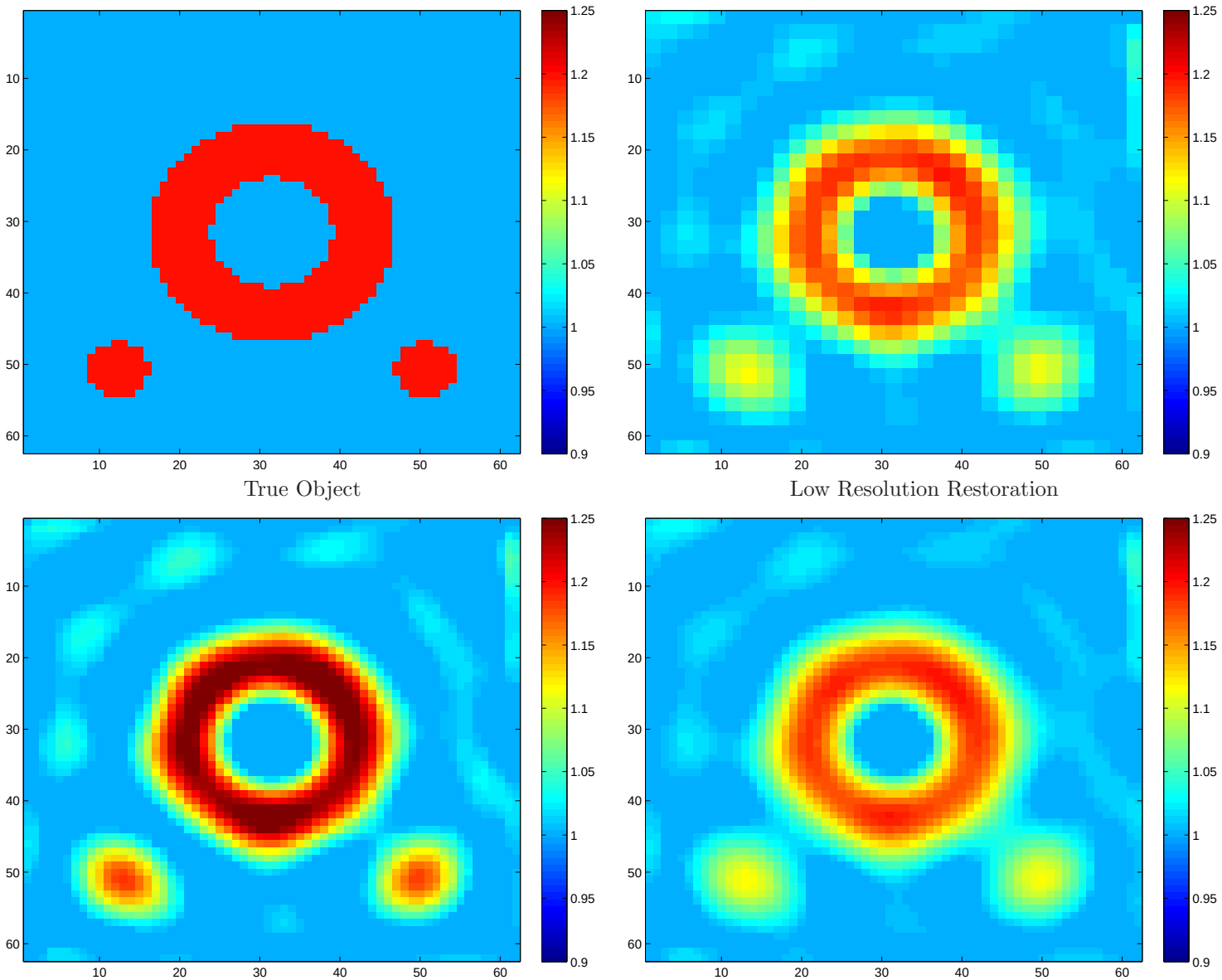
Gauss-Newton steps: 10.

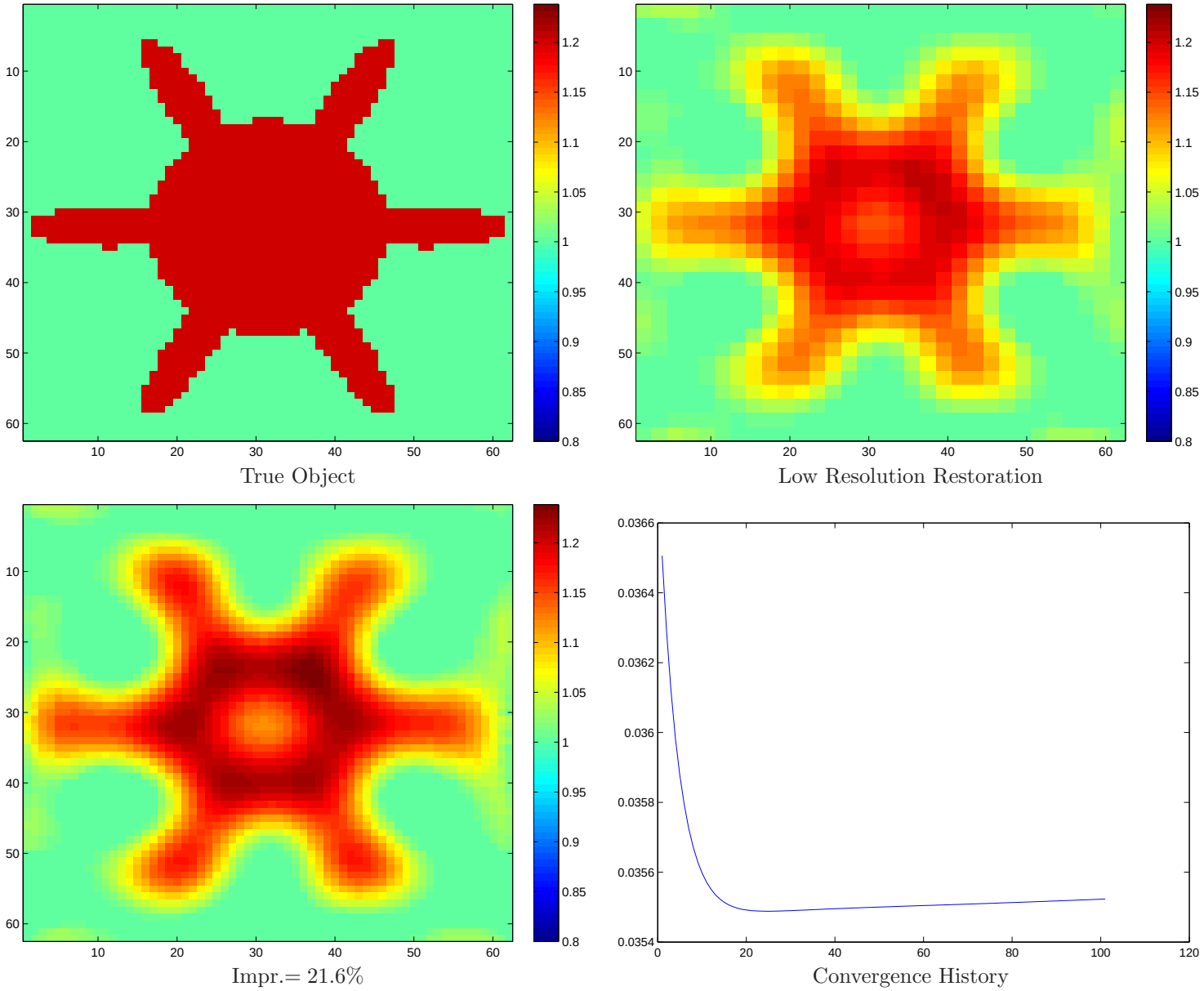


L- Two identical homogeneous circular cylinders of diameter $D = 0.2$ m and contrast $\chi = 1.1$, centered at $r_1 = (-0.25, 0.0)$ and $r_2 = (-0.25, 0.0)$, respectively. Restoration errors: 6.7%.

R- One circular homogeneous cylinder, diameter 0.6 m., contrast $\chi = 0.4$, with a square hole centered in $r = (-0.1, 0.1)$ and side 0.2 m. Restoration errors: 6.1%.

SuperResolution




True Object
Low Resolution Restoration
Impr.= 21.6%
Convergence History

Open Problems

• The blocks of the Fréchet are highly correlated (different rotations and illuminations). This is not yet considered in the solving scheme.

• Theoretically, it is not proved that iterative block splitting methods with inner regularization yield to regularization algorithms for nonlinear inverse problems.

• By an applicative point of view, the microwave imaging research is in a pioneer stage. Numerical results are still poor.

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