Multilinear Algebra Based Fitting of a Sum of Exponentials to Oversampled Data

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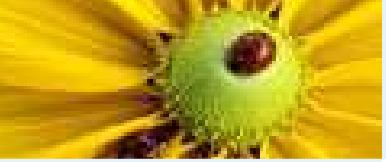

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  - The use of the complex exponential model;
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  - Standard Matrix approach;
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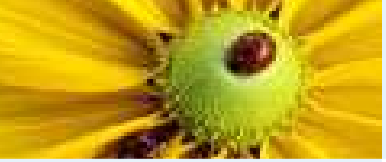

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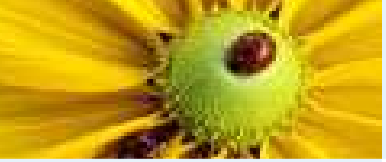

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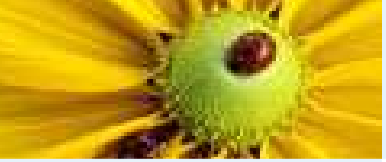

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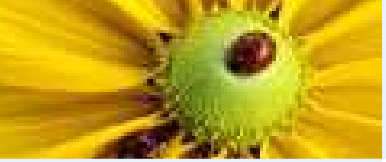

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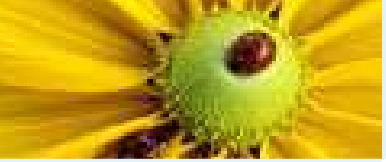

The use of the complex exponential model

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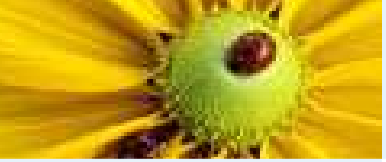
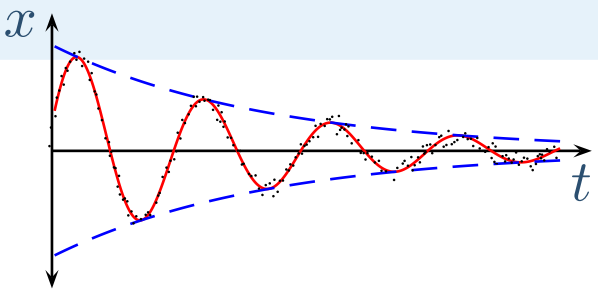
This model is ubiquitous in digital signal processing applications:

- Nuclear magnetic resonance (NMR) spectroscopy,
- audio processing,
- speech processing,
- material health monitoring,
- shape from moments



The complex exponential model



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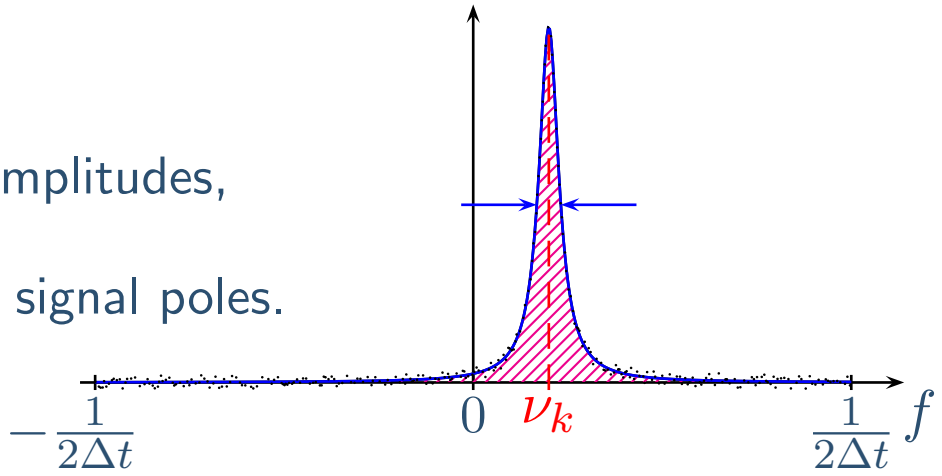
The discrete-time model has the following form:

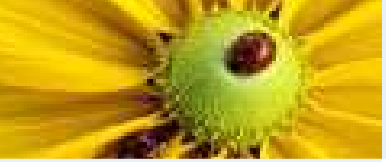
$$x_n = \sum_{k=1}^{K} a_k \exp\{j\varphi_k\} \exp\{(\alpha_k + 2j\pi\nu_k)n.\Delta t\} + b_n \quad n = 0, \ldots, N-1, \tag{1}$$

amplitude, phase, damping factor, frequency, sampling time interval, WGN

$$x_n = \sum_{k=1}^{K} c_k z_k^n + b_n \quad n = 0, \ldots, N-1, \tag{2}$$

- $c_k = a_k \exp\{j\varphi_k\}$: complex amplitudes,
- $z_k = \exp\{(\alpha_k + 2j\pi\nu_k)\Delta t\}$: signal poles.





Standard Matrix approach

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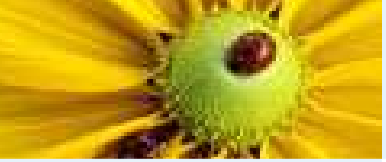
Starting point for a subspace representation $\Longrightarrow \{x_n\}$ is arranged in a Hankel matrix:

$$\boldsymbol{H} = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{M-1} \\ x_1 & x_2 & \ddots & \cdots & \vdots \\ x_2 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & x_{N-2} \\ x_{L-1} & \cdots & \cdots & x_{N-2} & x_{N-1} \end{pmatrix}$$



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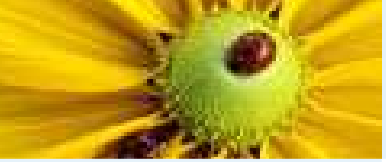
$\boldsymbol{H}$ can be factorized as follows (noise-free case):

$$\boldsymbol{H} = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^1 & \cdots & z_K^1 \\ z_1^2 & \cdots & z_K^2 \\ \vdots & \vdots & \vdots \\ z_1^{L-1} & \cdots & z_K^{L-1} \end{pmatrix} \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_K \end{pmatrix} \begin{pmatrix} 1 & z_1^1 & z_1^2 & \cdots & z_1^{M-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & z_K^1 & z_K^2 & \cdots & z_K^{M-1} \end{pmatrix}$$

$$= \underbrace{\boldsymbol{S}\boldsymbol{C}\boldsymbol{T}^{\mathrm{T}}}_{\text{rank-K matrix}} \quad (3)$$

$\Longrightarrow$ *Vandermonde decomposition*

Due to the structure of the noise-free model x_n, $\boldsymbol{H}$ is rank deficient
The rank equals the number of signal poles (model order)



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$$\boldsymbol{H} = \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ z_1^1 & \cdots & z_K^1 \\ z_1^2 & \cdots & z_K^2 \\ \vdots & \vdots & \vdots \\ z_1^{L-1} & \cdots & z_K^{L-1} \end{pmatrix}}_{\text{subspace-of-interest}} \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_K \end{pmatrix} \begin{pmatrix} 1 & z_1^1 & z_1^2 & \cdots & z_1^{M-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & z_K^1 & z_K^2 & \cdots & z_K^{M-1} \end{pmatrix}$$

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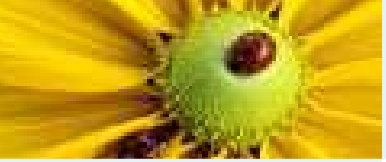

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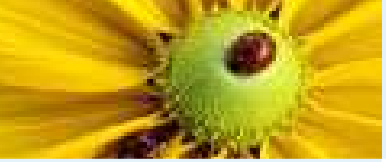
In the noise-free case, this rank deficiency is reflected by the SVD of $\boldsymbol{H}$:

$$\boldsymbol{H} = \begin{pmatrix} | & \cdots & | & \cdots & | \\ U_1 & \cdots & U_K & \cdots & U_L \\ | & \cdots & | & \cdots & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 & & \\ & \ddots & & & 0 \\ 0 & & \lambda_K & & \\ \hline & & & 0 & \\ & 0 & & & 0 \end{pmatrix} \begin{pmatrix} — & V_1^{\mathsf{H}} & — \\ \vdots & \vdots & \vdots \\ — & V_K^{\mathsf{H}} & — \\ \hline \vdots & \vdots & \vdots \\ — & V_M^{\mathsf{H}} & — \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\boldsymbol{U}} & \boldsymbol{U}_0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\Sigma}} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{V}}^{\mathsf{H}} \\ \boldsymbol{V}_0^{\mathsf{H}} \end{pmatrix} = \widehat{\boldsymbol{U}} \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{V}}^{\mathsf{H}}$$



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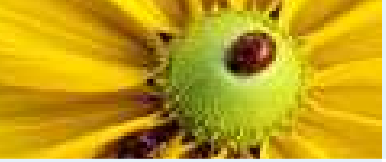
In the presence of noise $\boldsymbol{H}$ is a full rank matrix:

$$\boldsymbol{H} = \begin{pmatrix} | & \cdots & | & \cdots & | \\ U_1 & \cdots & U_K & \cdots & U_L \\ | & \cdots & | & \cdots & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & 0 & & 0 \\ & \ddots & & & \\ 0 & & \lambda_K & & \\ \hline 0 & & & & \boldsymbol{\Sigma}_0 \end{pmatrix} \begin{pmatrix} — & V_1^{\mathsf{H}} & — \\ \vdots & \vdots & \vdots \\ — & V_K^{\mathsf{H}} & — \\ \hline \vdots & \vdots & \vdots \\ — & V_M^{\mathsf{H}} & — \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\boldsymbol{U}} & \boldsymbol{U}_0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{\Sigma}} & 0 \\ 0 & \boldsymbol{\Sigma}_0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{V}}^{\mathsf{H}} \\ \boldsymbol{V}_0^{\mathsf{H}} \end{pmatrix} = \widehat{\boldsymbol{U}} \widehat{\boldsymbol{\Sigma}} \widehat{\boldsymbol{V}}^{\mathsf{H}}$$



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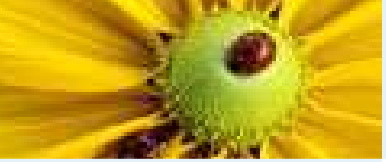
If the signal poles are close (closely spaced peaks), the Vandermonde vectors are almost dependent. Therefore, in the presence of noise, $\widehat{\boldsymbol{U}}$ might yield a very poor estimate of the column space of $\boldsymbol{S}$.



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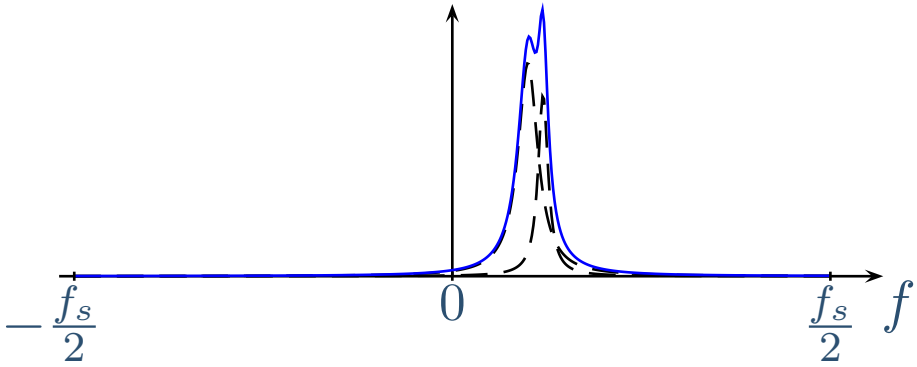

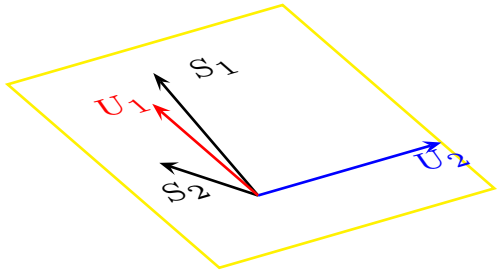

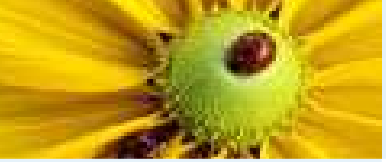

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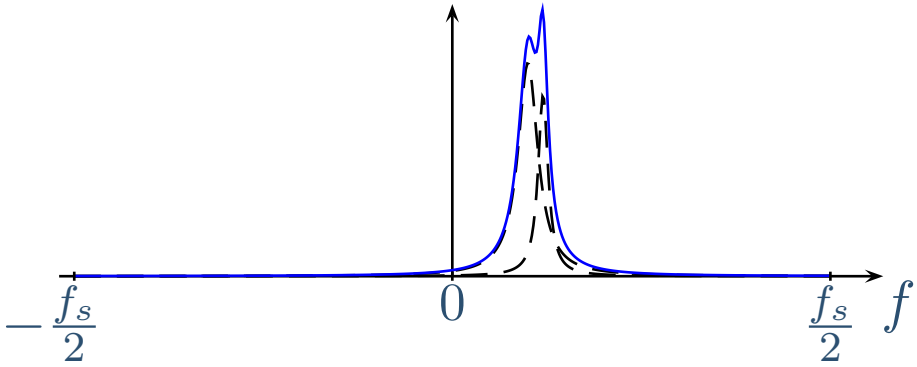
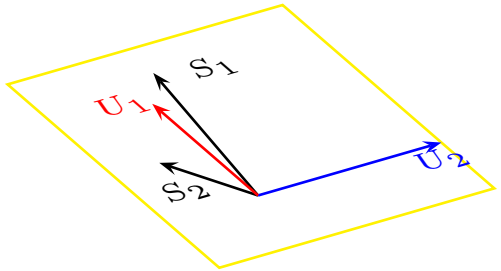
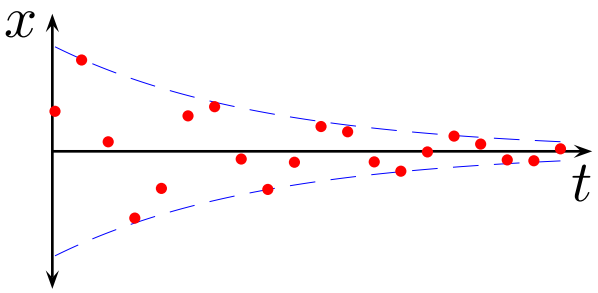
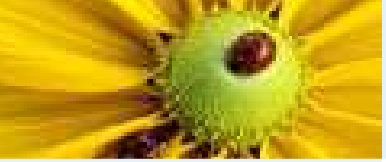




Oversampling yields higher accuracy





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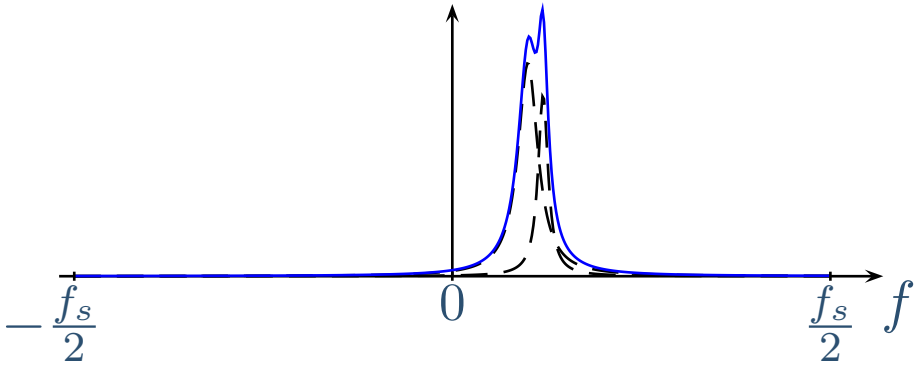

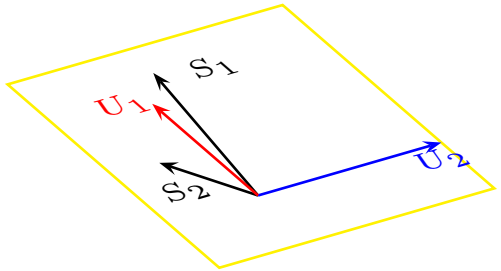

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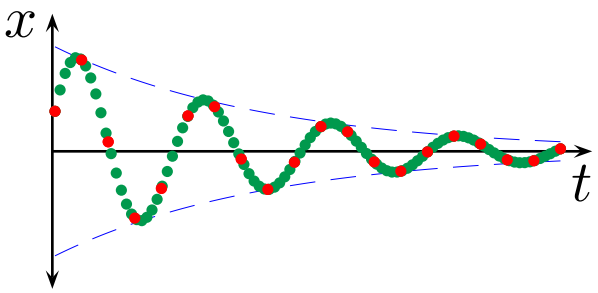

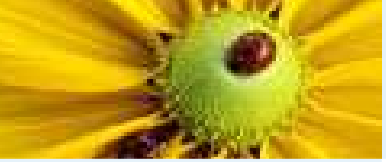

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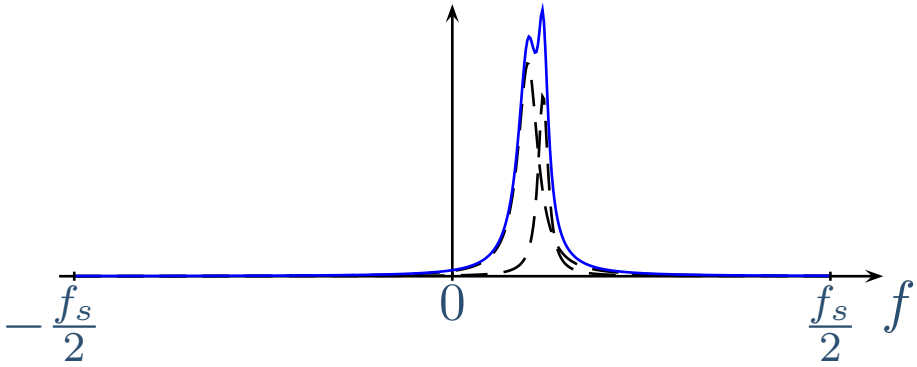

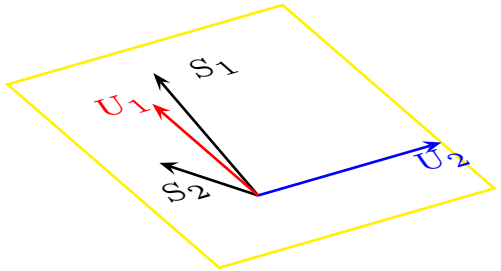

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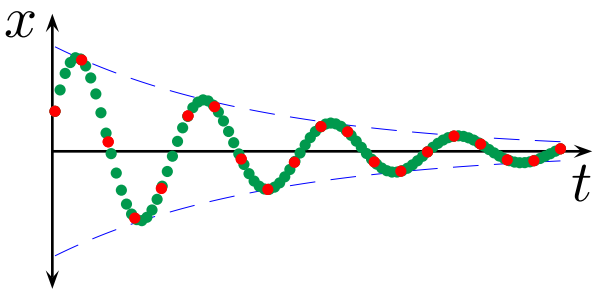
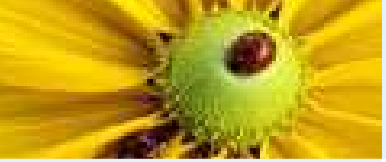


$\Longrightarrow$ The SVD of $\boldsymbol{H}$ becomes computationally expensive



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$$x_0 \quad x_1 \quad \cdots \quad x_{D-1} \quad x_D \quad x_{D+1} \quad \cdots \quad x_{2D-1} \quad x_{2D} \quad x_{2D+1} \quad \cdots \quad x_{N-1}$$

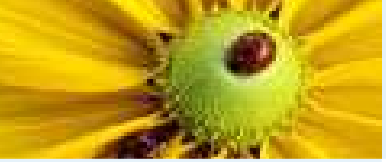
$$x_n^{(d)} = x_{nD+d} = \sum_{k=1}^{K} c_k z^{nD+d} \quad n = 0, \ldots, \frac{N}{D} - 1, \; d = 0, \ldots, D-1 \,,$$

$$\boldsymbol{H} = \left[\, \boldsymbol{H}_0 \mid \boldsymbol{H}_1 \mid \ldots \mid \boldsymbol{H}_{D-1} \, \right].$$

$$\boldsymbol{H} = \boldsymbol{S} \left[\, \boldsymbol{C}_0 \boldsymbol{T}^{\mathsf{T}} \mid \boldsymbol{C}_1 \boldsymbol{T}^{\mathsf{T}} \mid \ldots \mid \boldsymbol{C}_{D-1} \boldsymbol{T}^{\mathsf{T}} \, \right] = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{H}}$$

$$\boldsymbol{S} = \begin{pmatrix} 1 & \cdots & 1 \\ z_1^D & \cdots & z_K^D \\ z_1^{2D} & \cdots & z_K^{2D} \\ \vdots & \vdots & \vdots \\ z_1^{(L_D-1)D} & \cdots & z_K^{(L_D-1)D} \end{pmatrix}$$

Low computation cost
good accuracy



More structure

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The structure of $\boldsymbol{H}$ is more complex:

$$\boldsymbol{C}_d = \text{diag}\{c_1 \ldots, c_k\}.\,(\text{diag}\{z_1 \ldots, z_k\})^d$$

The results could be better if this structure could be taken into account.

Is any matrix decomposition able exploit the complete structure of $\boldsymbol{H}$?

$\Longrightarrow$ limits of the traditional linear algebra !!

The whole structure can be handled using multilinear algebra

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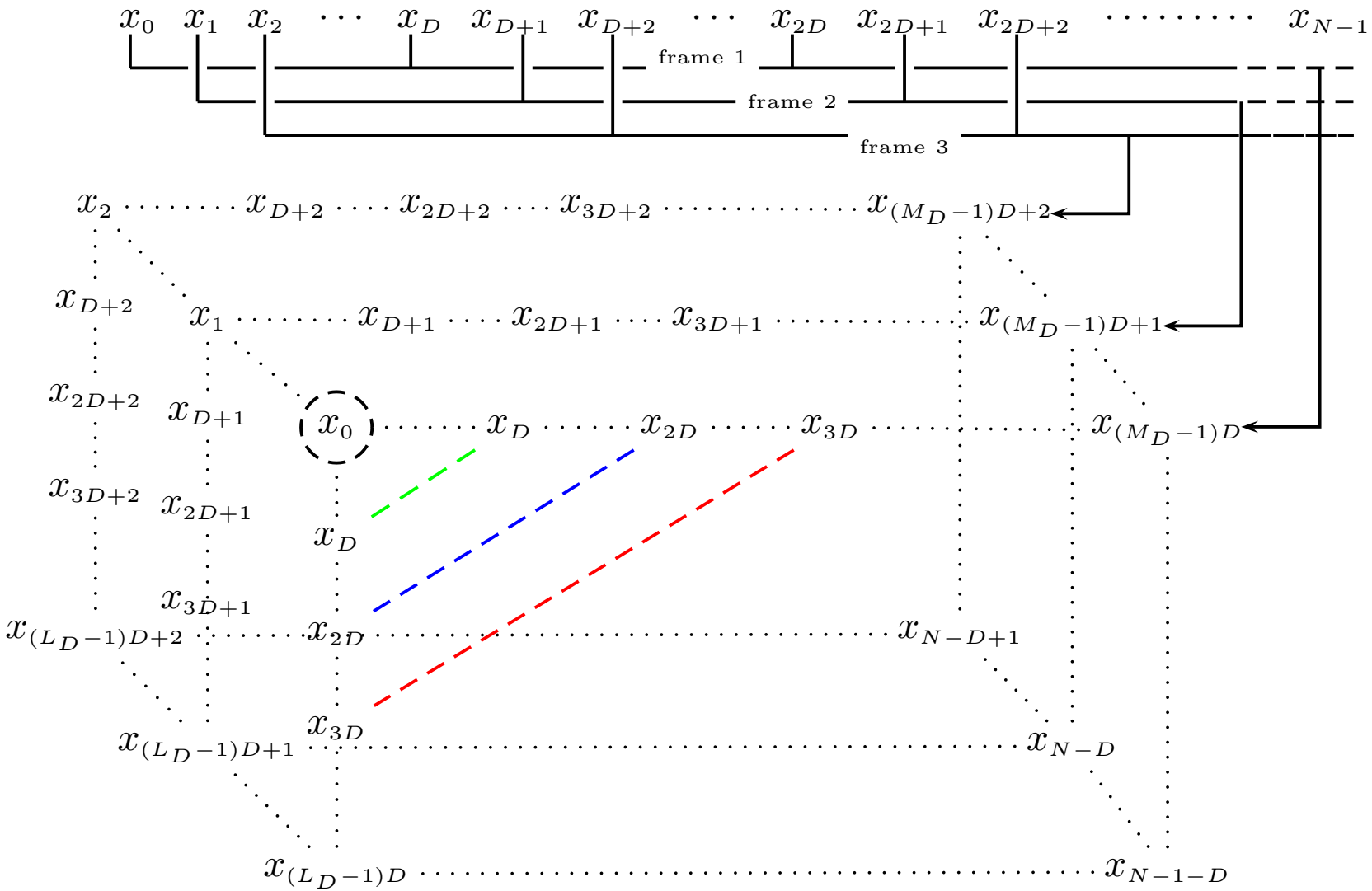

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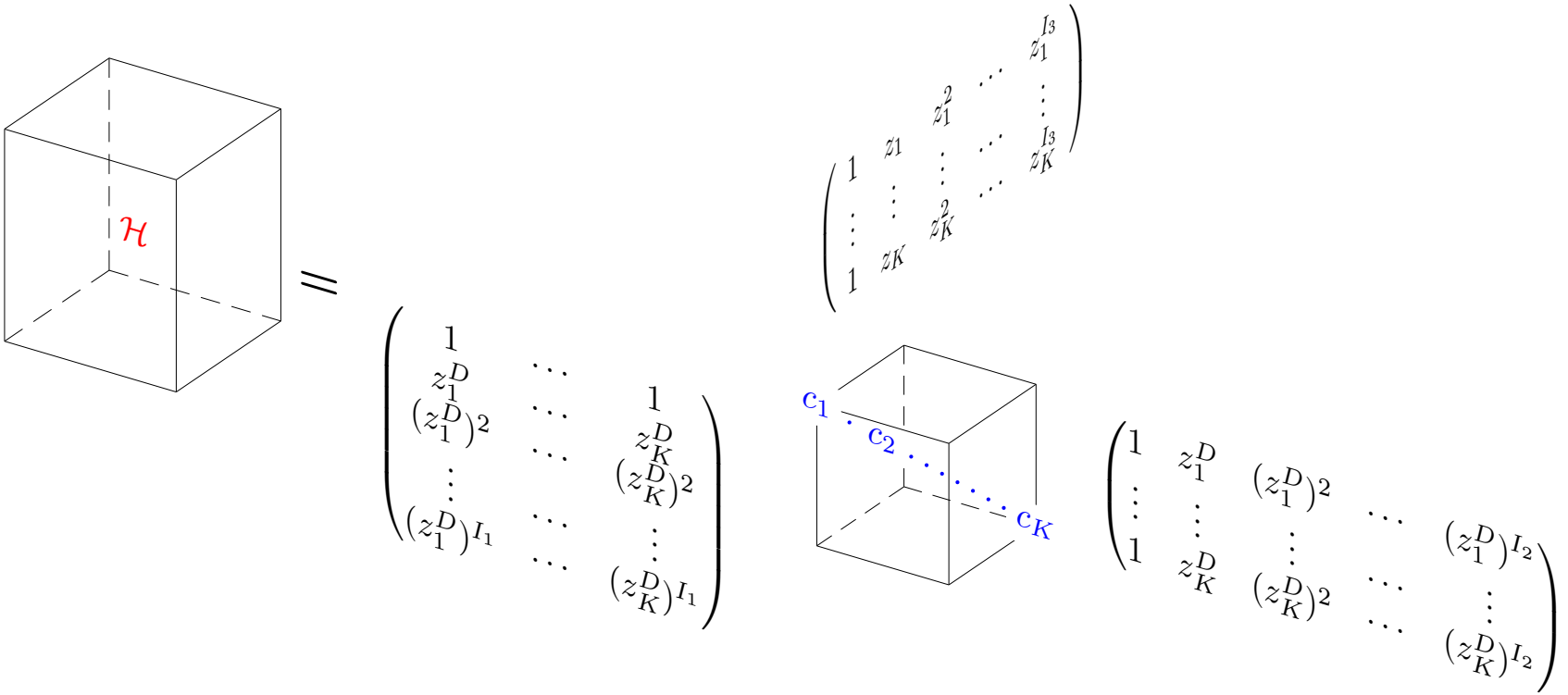

$$\mathcal{H} = \widehat{\mathcal{C}} \bullet_1 \widehat{\boldsymbol{S}}^{(1)} \bullet_2 \widehat{\boldsymbol{S}}^{(2)} \bullet_3 \widehat{\boldsymbol{S}}^{(3)}$$

In the noise-free case $\mathcal{H}$ is a rank-K tensor

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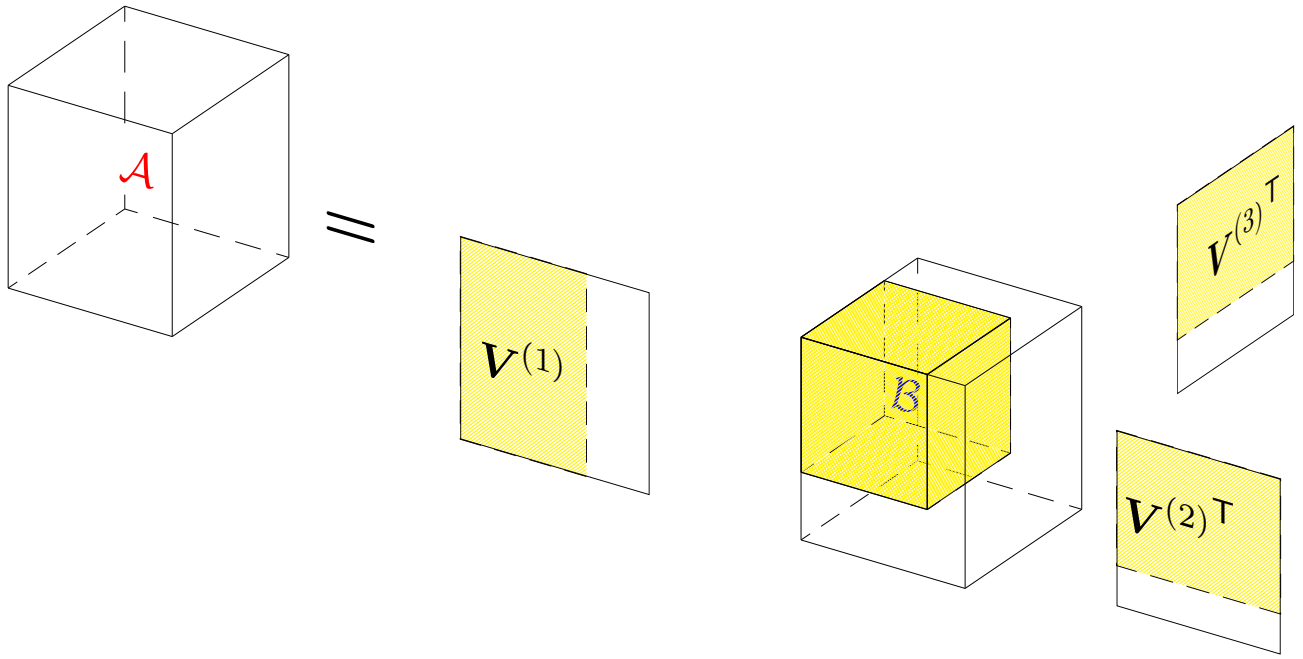
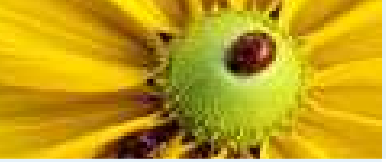
The Tucker Decomposition or Higher-Order SVD



If $\mathcal{H}$ is n-mode rank deficient, only the shaded part of the core-tensor contains entries different from zero.

$$\mathcal{H} = \widehat{\mathcal{S}} \bullet_1 \widehat{\boldsymbol{V}}^{(1)} \bullet_2 \widehat{\boldsymbol{V}}^{(2)} \bullet_3 \widehat{\boldsymbol{V}}^{(3)}$$

$\widehat{\ }$ denotes the (truncated) yellow part



Dimensionality reduction

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Given a complex third-order tensor $\mathcal{H} \in \mathbb{C}^{L \times M \times D}$, find a rank-$(K, K, K)$ tensor $\widehat{\mathcal{H}}$ that minimizes the least-squares cost function

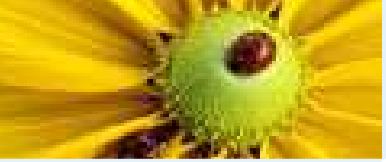
$$f(\widehat{\mathcal{H}}) = \left\| \mathcal{H} - \widehat{\mathcal{H}} \right\|^2. \tag{3}$$

Due to the n-rank constraints, $\widehat{\mathcal{H}}$ can be decomposed as :

$$\widehat{\mathcal{H}} = \mathcal{B} \bullet_1 \boldsymbol{U}^{(1)} \bullet_2 \boldsymbol{U}^{(2)} \bullet_3 \boldsymbol{U}^{(3)} \tag{4}$$

in which $\boldsymbol{U}^{(1)} \in \mathbb{C}^{L \times K}$, $\boldsymbol{U}^{(2)} \in \mathbb{C}^{M \times K}$, $\boldsymbol{U}^{(3)} \in \mathbb{C}^{D \times K}$ each have orthonormal columns and $\mathcal{B} \in \mathbb{C}^{K \times K \times K}$ is an all-orthogonal tensor.

HOOI [Kroonenberg '84, De Lathauwer '00], Newton [Eldén and Savas '08], Quasi-Newton [Savas and Lim '08], Conjugate gradient [Ishteva '08], Trust region [Ishteva '08], Oja [Ishteva '08], Krylov [Savas and Eldén '08]



Ill-conditioning issue

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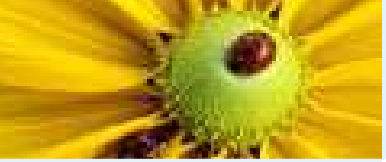
- The theory gives us the *true* rank of the data tensor,
- Since there is no decimation effect along the 3rd mode, the mode-3 subspace is generally ill-conditioned



Unsymmetric tensor approximation

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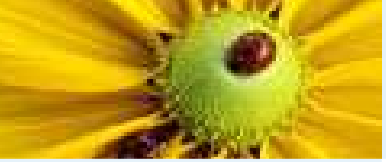
Theorem (Unsymmetric tensor approximation) *Consider a tensor* $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ *that is rank-*(R_1, R_2, R_3)*. Let the HOSVD of* $\mathcal{A}$ *be given by*

$$\mathcal{A} = \mathcal{B} \bullet_1 \boldsymbol{V}^{(1)} \bullet_2 \boldsymbol{V}^{(2)} \bullet_3 \boldsymbol{V}^{(3)}.$$

Then the best rank-$(R_1, R_2, \tilde{R}_3)$ *approximation of* $\mathcal{A}$*, with* $\tilde{R}_3 < R_3$*, is obtained by truncation of* $\mathcal{B}$ *and* $\boldsymbol{U}^{(3)}$*.*

- As a consequence of this theorem one can concentrate on the dominant part of the data tensor by decreasing the mode-3 rank without losing the data structure in the mode-1 and 2 subspace.



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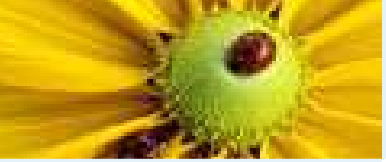
Decimative matrix approach:

$$x_n \longrightarrow x_m^{(d)} \longrightarrow \boldsymbol{H} = \left[\boldsymbol{H_0} \mid \boldsymbol{H_1} \mid \cdots \mid \boldsymbol{H_{D-1}} \right] \quad \begin{matrix} \boldsymbol{H_n} \in \mathbb{C}^{L \times M} \\ \boldsymbol{H} \in \mathbb{C}^{L \times M.D} \end{matrix}$$

$$\longrightarrow \boldsymbol{H} \overset{SVD}{=} \boldsymbol{U \Sigma V}^{\mathrm{H}} \longrightarrow \widehat{\boldsymbol{U}} = \boldsymbol{U}[:, 1:K]$$



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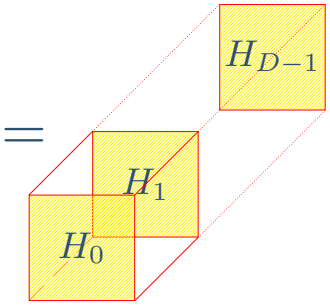
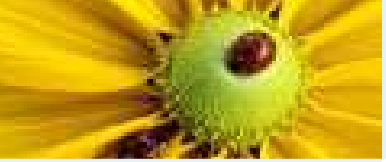


$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\mathcal{H} \in \mathbb{C}^{L \times M \times D}$$

$$\widehat{\mathcal{H}} = \mathcal{B} \bullet_1 \boldsymbol{U}^{(1)} \bullet_2 \boldsymbol{U}^{(2)} \bullet_3 \boldsymbol{U}^{(3)}$$

Best rank-(K, K, K') approx. of $\mathcal{H}$

$$K' \leqslant K$$



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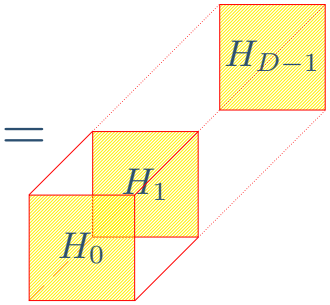
$$\longrightarrow \boldsymbol{H} \overset{SVD}{=} \boldsymbol{U \Sigma V}^{\mathrm{H}} \longrightarrow \widehat{\boldsymbol{U}} = \boldsymbol{U}[:, 1:K]$$

Decimative tensor approach:

$$x_n \longrightarrow x_m^{(d)} \longrightarrow \mathcal{H} =$$



$$\boldsymbol{H_n} \in \mathbb{C}^{L \times M}$$
$$\mathcal{H} \in \mathbb{C}^{L \times M \times D}$$

$$\widehat{\mathcal{H}} = \mathcal{B} \bullet_1 \boldsymbol{U}^{(1)} \bullet_2 \boldsymbol{U}^{(2)} \bullet_3 \boldsymbol{U}^{(3)}$$ Best rank-(K, K, K') approx. of $\mathcal{H}$

$K' \leqslant K$

Total least squares (TLS) solution:

$$[\widehat{\boldsymbol{U}}_{\downarrow} \widehat{\boldsymbol{U}}^{\uparrow}] = \boldsymbol{Y}_{(L-1)\times(L-1)} \boldsymbol{\Gamma} \boldsymbol{W}^{\mathrm{H}}_{2K \times 2K}$$

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_{11} & \boldsymbol{W}_{12} \\ \boldsymbol{W}_{21} & \boldsymbol{W}_{22} \end{bmatrix}$$

$$\Longrightarrow \boxed{\widehat{\widehat{\boldsymbol{Z}}} = -\boldsymbol{W}_{12} \boldsymbol{W}_{22}^{-1}}$$

Two closely spaced peaks, damped signal

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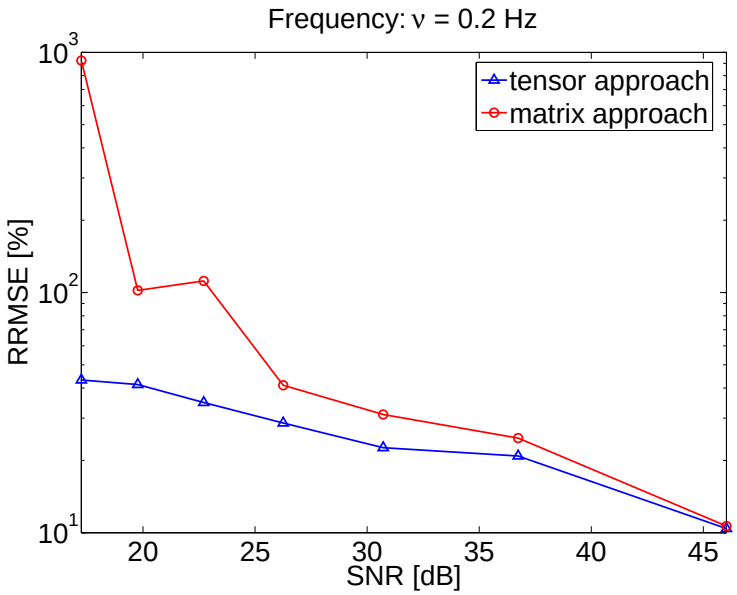
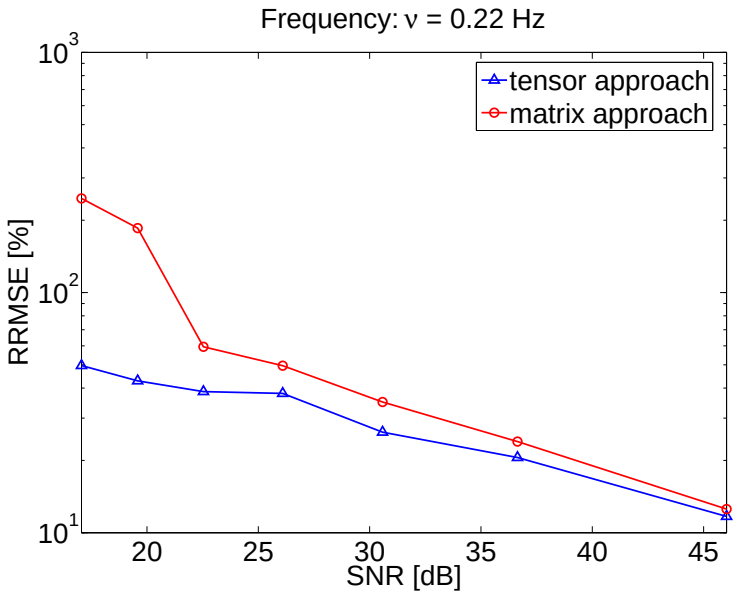
$x_n = \exp\{(-0.01 + 2j\pi 0.2)\Delta t.n\} + \exp\{(-0.02 + 2j\pi 0.22)\Delta t.n\}$, with $n = 0, \ldots, N-1$.

Number of samples $N = 625$, decimation factor $D = 25$, sampling time interval $\Delta t = 0.04$.

Tensor: $\mathcal{H}_{13\times 13\times 25} = \mathcal{B}_{2\times 2\times 2} \bullet_1 \boldsymbol{U}^{(1)}_{2\times 13} \bullet_2 \boldsymbol{U}^{(2)}_{2\times 13} \bullet_3 \boldsymbol{U}^{(3)}_{2\times 25}$

Matrix: $H_{13\times 325} = \widehat{\boldsymbol{\Sigma}}_{2\times 2} \bullet_1 \widehat{\boldsymbol{U}}_{2\times 13} \bullet_2 \widehat{\boldsymbol{V}}^*_{2\times 325}$ $(= \widehat{\boldsymbol{U}}\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{V}}^{\mathsf{H}})$





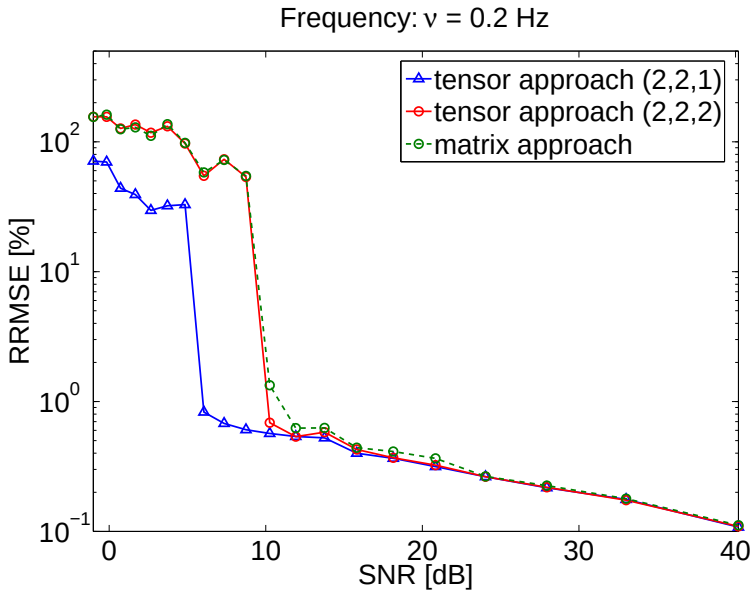
Two closely spaced peaks, undamped signal

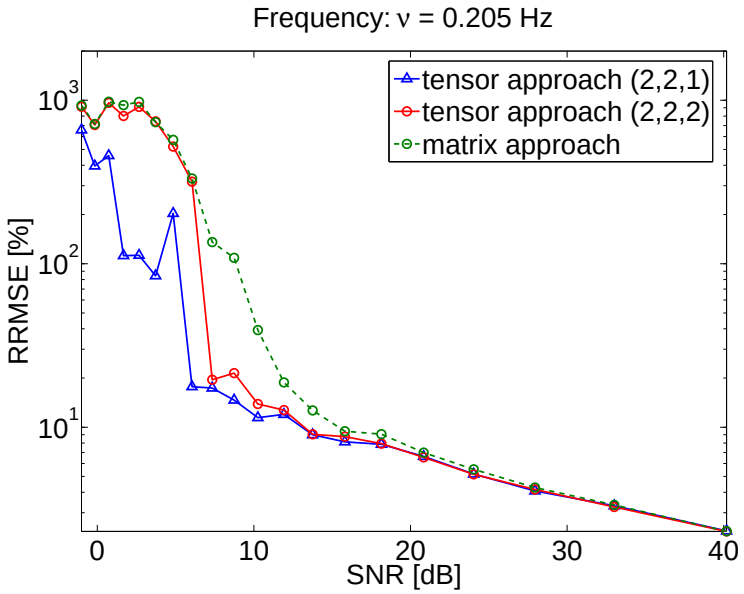
$x_n = \exp\{(2j\pi 0.2)\Delta t.n\} + \exp\{(2j\pi 0.205)\Delta t.n\}$, with $n = 0, \ldots, N-1$.

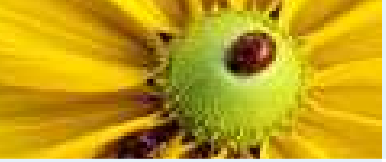
Number of samples $N = 1000$, decimation factor $D = 25$, sampling time interval $\Delta t = 0.1$.

Tensor: $\mathcal{H}_{50\times 50\times 10} = \mathcal{B}_{2\times 2\times 2/1} \bullet_1 \boldsymbol{U}^{(1)}_{2\times 50} \bullet_2 \boldsymbol{U}^{(2)}_{2\times 50} \bullet_3 \boldsymbol{U}^{(3)}_{2/1\times 10}$

Matrix: $H_{50\times 500} = \widehat{\boldsymbol{\Sigma}}_{2\times 2} \bullet_1 \widehat{\boldsymbol{U}}_{2\times 50} \bullet_2 \widehat{\boldsymbol{V}}^*_{2\times 500}$ $(= \widehat{\boldsymbol{U}}\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{V}}^{\mathsf{H}})$







Five-peak, damped signal 1/3

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Peak k	ν_k [Hz]	α_k [s^{-1}]	a_k [a.u.]^a	φ_k [°]^b
1	-1379	208	6.1	15
2	-685	256	9.9	15
4	-271	197	6.0	15
3	353	117	2.8	15
5	478	808	17	15

^{a} a.u., arbitrary units

^{b} $\varphi_k \times \frac{\pi}{180}$ expresses the phase in radians

Tensor:

$$\mathcal{H}_{26\times25\times10} = \mathcal{B}_{2\times2\times1/2/3/4/5} \bullet_1 \boldsymbol{U}^{(1)}_{5\times26} \bullet_2 \boldsymbol{U}^{(2)}_{5\times25} \bullet_3 \boldsymbol{U}^{(3)}_{1/2/3/4/5\times10}$$

Matrix: $H_{26\times250} = \widehat{\boldsymbol{\Sigma}}_{5\times5} \bullet_1 \widehat{\boldsymbol{U}}_{5\times26} \bullet_5 \widehat{\boldsymbol{V}}^*_{5\times250}$ $(= \widehat{\boldsymbol{U}}\widehat{\boldsymbol{\Sigma}}\widehat{\boldsymbol{V}}^{\mathsf{H}})$

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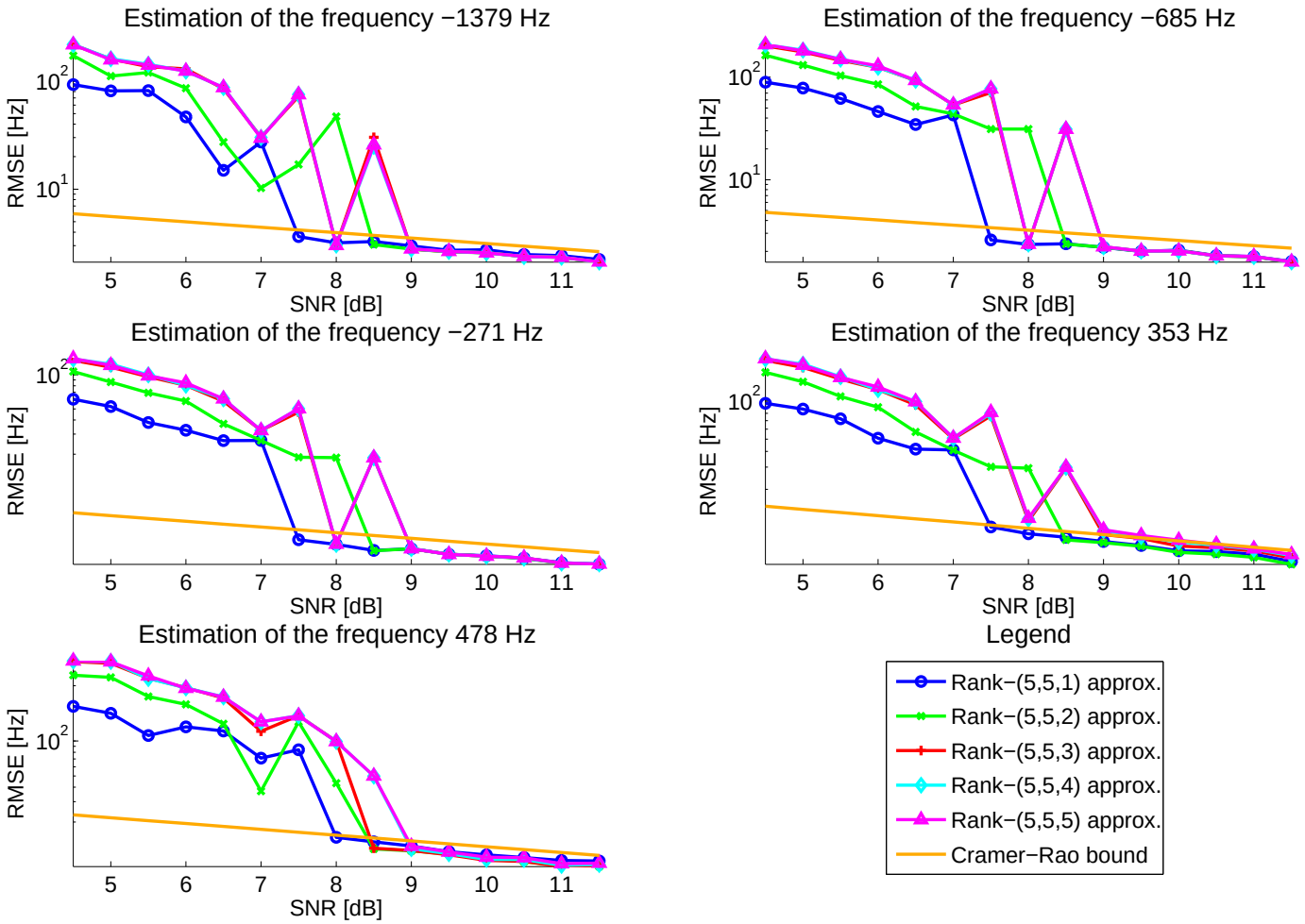
Decreasing of the mode-3 rank from 5 to 1:



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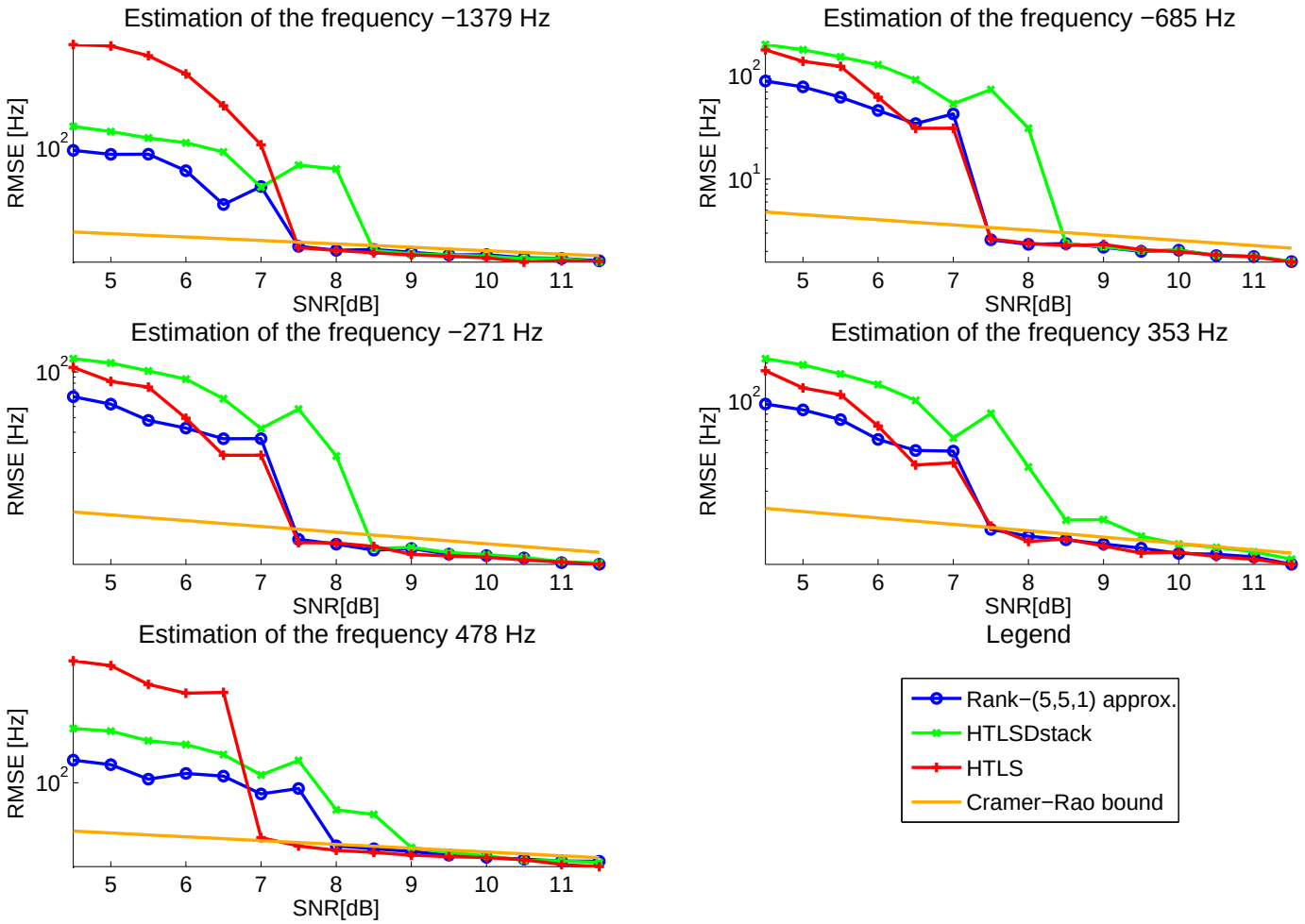
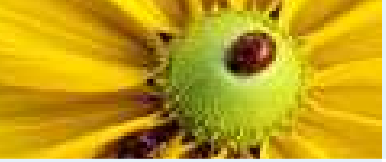
Performance comparison to matrix algorithms:





Conclusions

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- We considered oversampled signals whose poles are potentially very close,
- In the decimative matrix approach the structure is partially taken into account,
- A multilinear approach helps to reveal the rest of the structure (e.g. HOVDMD of the $(L \times M \times D)$-tensor),
- And shows a rank deficiency: $\mathcal{H}$ is a rank-(K, K, K) tensor,
- Ill-conditioned modes can be treated in a different manner than well-conditioned modes (unsymmetric dimensionality reduction),
- The best rank-(K, K, K') with $K' \leqslant K$ approximation of the noisy data tensor consistently yields a better subspace estimate than the best rank-K approximation of the noisy data matrix.