Classifications of quasiseparable matrices in terms of recurrence relations

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Orthogonal Polynomials Related to Structured Matrices

Moment Matrices

- **Hankel matrices.** Defined by $\mathcal{O}(n)$ parameters $\{h_k\}$.

$$H = \left[\ h_{k+j}\ \right] = \begin{bmatrix} h_0 & \mathbf{h_1} & h_2 & \cdots & h_{n-1} \\ \mathbf{h_1} & h_2 & & \iddots & \vdots \\ h_2 & & \iddots & & h_{2n-3} \\ \vdots & \iddots & & h_{2n-3} & \mathbf{h_{2n-2}} \\ h_{n-1} & \cdots & h_{2n-3} & \mathbf{h_{2n-2}} & h_{2n-1} \end{bmatrix}$$

- **Toeplitz matrices.** Defined by $\mathcal{O}(n)$ parameters $\{t_k\}$.

$$C = \left[\ t_{k-j}\ \right] = \begin{bmatrix} t_0 & \mathbf{t_{-1}} & \cdots & \cdots & t_{-n+1} \\ t_1 & t_0 & \mathbf{t_{-1}} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & t_0 & \mathbf{t_{-1}} \\ t_{n-1} & \cdots & \cdots & t_1 & t_0 \end{bmatrix}$$

Orthogonal Polynomials Related to Structured Matrices

Moment Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- For a given inner product, the **moment matrix** is

$$M = [\langle x^k, x^j \rangle] = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle & \langle 1, x^2 \rangle & \ldots & \langle 1, x^n \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle & \langle 1, x^2 \rangle & \ldots & \langle x, x^n \rangle \\ \langle x^2, 1 \rangle & \langle x^2, x \rangle & \langle x^2, x^2 \rangle & \ldots & \langle x^2, x^n \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ \langle x^n, 1 \rangle & \langle x^n, x \rangle & \langle x^n, x^2 \rangle & \ldots & \langle x^n, x^n \rangle \end{bmatrix}$$

- For an inner product defined by integration on the real line,

$$\langle p(x), q(x) \rangle = \int_a^b p(x)q(x)w^2(x)dx, \quad \Rightarrow \quad \langle x^k, x^j \rangle = \int_a^b x^{(k+j)} w^2(x)dx,$$

  and M is Hankel.

- Hankel matrices are related to real–orthogonal polynomials.

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- For an inner product defined by integration on the unit circle,

$$\langle p(x), q(x) \rangle = \int_{-\pi}^{\pi} p(e^{i\theta}) \cdot \overline{q(e^{i\theta})} w^2(\theta) d\theta \Rightarrow \langle x^k, x^j \rangle = \int_{-\pi}^{\pi} x^{(k-j)} w^2(\theta) d\theta,$$

and M is Toeplitz.

- Toeplitz matrices are related to Szegö polynomials.

Orthogonal Polynomials Related to Structured Matrices

Recurrent Matrices

- **Tridiagonal matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

- **Unitary Hessenberg matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$U = \begin{bmatrix} -\rho_1 \rho_0{}^* & -\rho_2 \mu_1 \rho_0{}^* & \cdots & -\rho_n \mu_{n-1} \ldots \mu_1 \rho_0{}^* \\ \mu_1 & -\rho_2 \rho_1{}^* & \cdots & -\rho_n \mu_{n-1} \ldots \mu_2 \rho_1{}^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n \mu_{n-1} \rho_{n-2}{}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n \rho_{n-1}{}^* \end{bmatrix}$$

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Recurrent Matrices

- Both of these classes of matrices are related to **orthogonal polynomials**.
- The system of polynomials defined by $\mathbf{r_k}(x) = \det(xI - T)_{(\mathbf{k}\times\mathbf{k})}$ where

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

are real–orthogonal polynomials.

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- Both of these classes of matrices are related to **orthogonal polynomials**.
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$$U = \begin{bmatrix} -\rho_1{\rho_0}^* & -\rho_2\mu_1{\rho_0}^* & \cdots & -\rho_n\mu_{n-1}...\mu_1{\rho_0}^* \\ \mu_1 & -\rho_2{\rho_1}^* & \cdots & -\rho_n\mu_{n-1}...\mu_2{\rho_1}^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n\mu_{n-1}{\rho_{n-2}}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n{\rho_{n-1}}^* \end{bmatrix}$$

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Generalizations of these Structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	?????????????????????????????

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Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	quasiseparable matrices

Quasiseparable Matrices

- **Definition.** A matrix C is (H,m)**–quasiseparable** if it is strongly upper Hessenberg (nonzero subdiagonals, zeros below that) and

$$\max \text{Rank} C_{12} = m$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline & * \end{array}\right]$$

- **Previous work.** Chandrasekaran, Eidelman, Fasino, Gemignani, Gohberg, Gu, Kailath, Koltracht, Mastronardi, Olshevsky, Van Barel, Vandebril...

- A system of polynomials related to an (H,m)–quasiseparable matrix C as **characteristic polynomials** of principal submatrices of C, i.e.

$$r_k(x) = \det(xI - C_{k\times k})$$

will be called (H,m)**–quasiseparable polynomials.**

Important Special Cases of Quasiseparable Matrices

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k\times k})$ associated with C are **real orthogonal polynomials** with recurrence relations

$$r_k(x) = \frac{1}{q_k}(x - d_k) r_{k-1}(x) - \frac{g_{k-1}}{q_k} r_{k-2}(x)$$

- The matrix C is $(H,1)$**–quasiseparable**.

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Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k\times k})$ associated with C are the **Szegö polynomials** with recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} G_{k-1}(x) \\ xr_{k-1}(x) \end{bmatrix}$$

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\mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\
\hline
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0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\
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The Difference Between These Motivating Examples

- **Unitary Hessenberg matrices.**

$$\begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 \end{bmatrix}$$

- **Tridiagonal matrices.**

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

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- **Unitary Hessenberg matrices.** strictly upper triangular part is part of a low rank matrix.

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- **Tridiagonal matrices.** strictly upper triangular part is NOT part of a low rank matrix.

$$C = \begin{bmatrix} & g_1 & 0 & 0 & 0 \\ & & g_2 & 0 & 0 \\ & & & g_3 & 0 \\ & & & & g_4 \\ & & & & \end{bmatrix}$$

Semiseparable matrices

- **Definition.** A matrix R is called (r_L, r_U)–semiseparable if for some r_L, r_U we have

$$R = D + tril(R_L) + triu(R_U),$$

where rank$R_L = r_L$, rank$R_U = r_U$, with some R_L, R_U.

- **Example.** $(1,1)$–semiseparable:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

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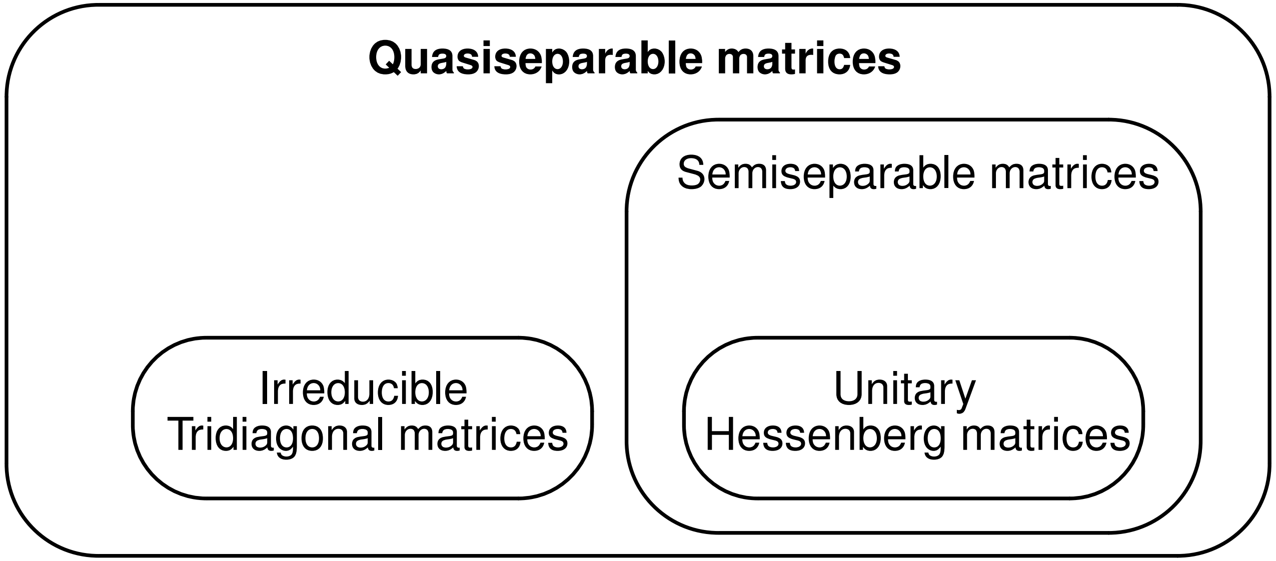
$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Quasiseparable, Semiseparable, and Subclasses



Quasiseparable matrices

Semiseparable matrices

Irreducible
Tridiagonal matrices

Unitary
Hessenberg matrices

Generator Representation of an $(H,1)$–Quasiseparable Matrix

$$\begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

Generator Representation of an $(H,1)$–Quasiseparable Matrix

$$\begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix}$$

➠ This **generator representation** exists for any $(H,1)$**–quasiseparable** matrix.

Classification of $(H, 1)$–quasiseparable matrices in terms of recurrence relations

Joint work with Yuli Eidelman, Israel Gohberg, and Vadim Olshevsky

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

$$\mathbf{r_k}(x) = \det(xI - A)_{(\mathbf{k}\times\mathbf{k})}$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

$$\mathbf{r_k}(x) = \mathbf{x} \cdot \mathbf{r_{k-1}}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

$$\mathbf{r_k}(x) = 2\mathbf{x} \cdot \mathbf{r_{k-1}}(x) - \mathbf{r_{k-2}}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

$$\mathbf{r_k}(x) = (\alpha_k \mathbf{x} - \delta_k)\mathbf{r_{k-1}}(x) - \gamma_k \mathbf{r_{k-2}}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations (2-term)

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r_{k+1}}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^{*} \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{x}\mathbf{r_k}(x) \end{bmatrix}$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations (3-term)

$$\mathbf{r_k}(x) = \left(\frac{1}{\mu_k}x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right)\mathbf{r_{k-1}}(x) - \left(\frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k}\cdot x\right)\mathbf{r_{k-2}}(x)$$

Efficient Recurrence Relations for Quasiseparable Polynomials

Matrices A	Polynomials $r_k(x)$
Lower shift matrix	Monomials
Tridiagonal matrix	Chebyshev polynomials
Tridiagonal matrix	Real–orthogonal polynomials
Unitary Hessenberg matrix	Szegö polynomials
Quasiseparable matrix	Quasiseparable polynomials

Recurrence relations

????????????????????????

Three-term Recurrence Relations.

Consider the class of polynomials satisfying more general three–term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k \, x + \gamma_k) r_{k-2}(x)$$

- Real-orthogonal polynomials: $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k \, r_{k-2}(x)$$

- Szegö polynomials (orthogonal on the unit circle): $\gamma_k = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) r_{k-2}(x)$$

Three-term Recurrence Relations.

Consider the class of polynomials satisfying more general three–term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\boxed{\beta_k}\, x + \boxed{\gamma_k}) r_{k-2}(x)$$

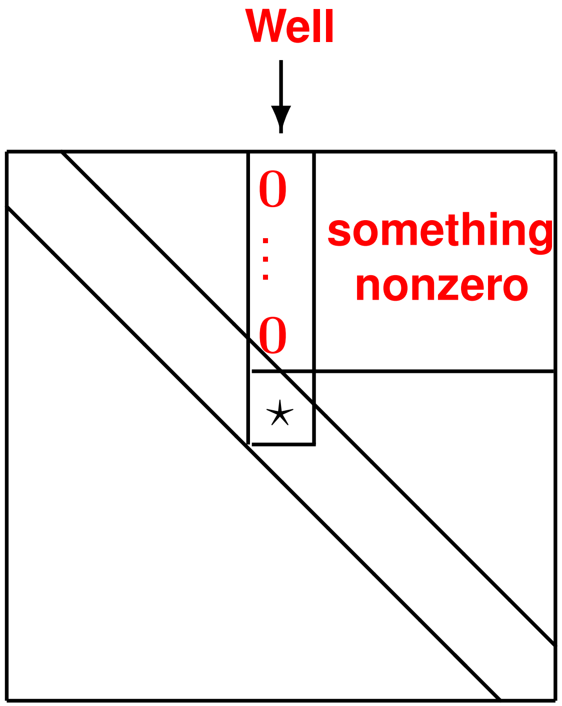
- Real-orthogonal polynomials: $\boxed{\beta_k} = 0$

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \boxed{\gamma_k}\, r_{k-2}(x)$$

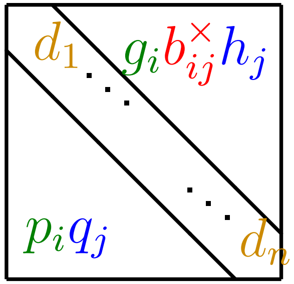
- Szegö polynomials (orthogonal on the unit circle): $\boxed{\gamma_k} = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\boxed{\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k}} \cdot x \right) r_{k-2}(x)$$

The Corresponding Matrix Class: Well–Free Matrices.



Well-Free Matrices & 3-term Recurrence Relations



Well-free

$(H,1)$–quasiseparable matrix

$\Leftrightarrow$ **equivalence**

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k x + \gamma_k) r_{k-2}(x)$$

3-term

recurrence relations

$$\{p_k, q_k, d_k, g_k, b_k, h_k\}$$

$$\mathbf{h_k \neq 0}$$

Quasiseparable generators

$\Leftrightarrow$ **conversions**

$$\{\alpha_k, \beta_k, \gamma_k, \delta_k\}$$

Recurrence relation coefficients

Subclasses of $(H,1)$–Quasiseparable Matrices

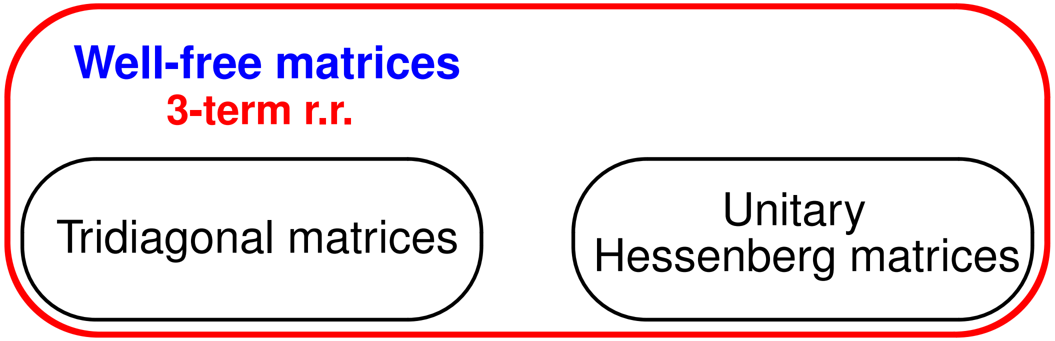
Corresponding recurrence relations

Tridiagonal matrices

Unitary
Hessenberg matrices

Subclasses of $(H,1)$–Quasiseparable Matrices

Corresponding recurrence relations



Well-free matrices

3-term r.r.

Tridiagonal matrices

Unitary
Hessenberg matrices

Szegö–type Two–term Recurrence Relations

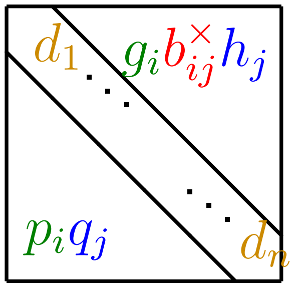
- Szegö polynomials satisfy two–term recurrence relations of the form

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r_{k+1}}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho^*_{k+1} \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{x}\mathbf{r_k}(x) \end{bmatrix}.$$

- Is there a class of polynomials **larger** than Szegö that satisfy two–term recurrence relations of the form

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r_{k+1}}(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_k(x) \\ (\delta_k x + \theta_k)\mathbf{r_k}(x) \end{bmatrix}?$$

Semiseparable Matrices & Szegö-type 2-term Recurrence Relations



$(H,1)$–semiseparable matrix

⇔
equivalence

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1} \\ (\delta_k x + \theta_k) r_{k-1} \end{bmatrix}$$

Szegö-type recurrence relations

$$\{p_k, q_k, d_k, g_k, b_k, h_k\}$$

$$\mathbf{b_k \neq 0}$$

Quasiseparable generators

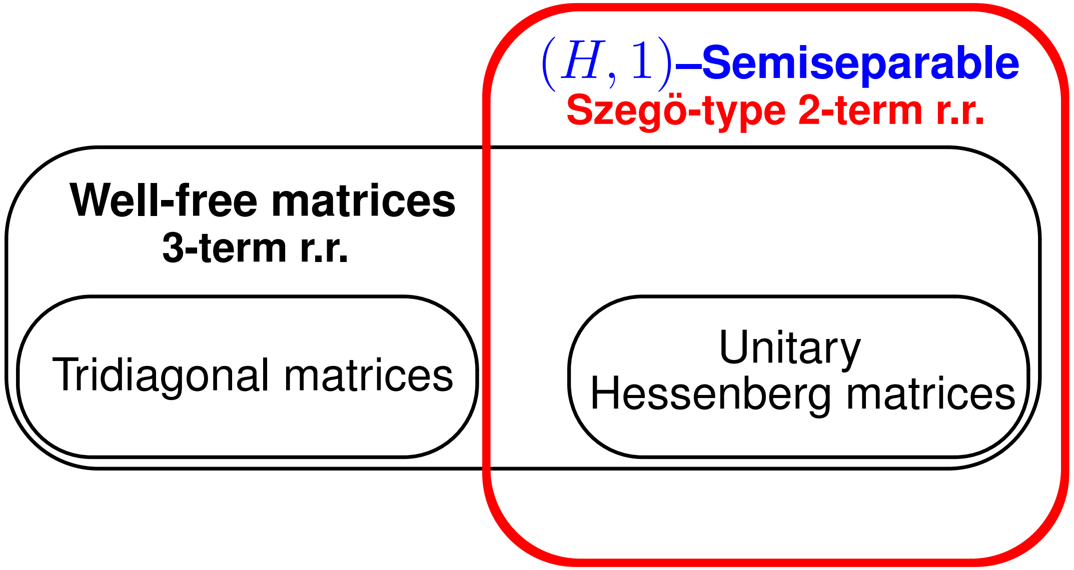
⇔
conversions

$$\{\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k\}$$

Recurrence relation coefficients

Subclasses of $(H,1)$–Quasiseparable Matrices

Corresponding recurrence relations



$(H,1)$–Semiseparable
Szegö-type 2-term r.r.

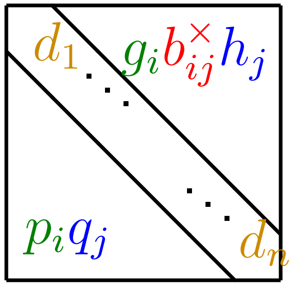
Well-free matrices
3-term r.r.

Tridiagonal matrices

Unitary
Hessenberg matrices

Quasiseparable Matrices & [EGO05]-type 2-term Recurrence Relations

A Complete Characterization of $(H,1)$–Quasiseparable Matrices



$(H,1)$–quasiseparable matrix

⇔
equivalence

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

[EGO05]-type recurrence relations

$$\{p_k, q_k, d_k, g_k, b_k, h_k\}$$

Quasiseparable generators

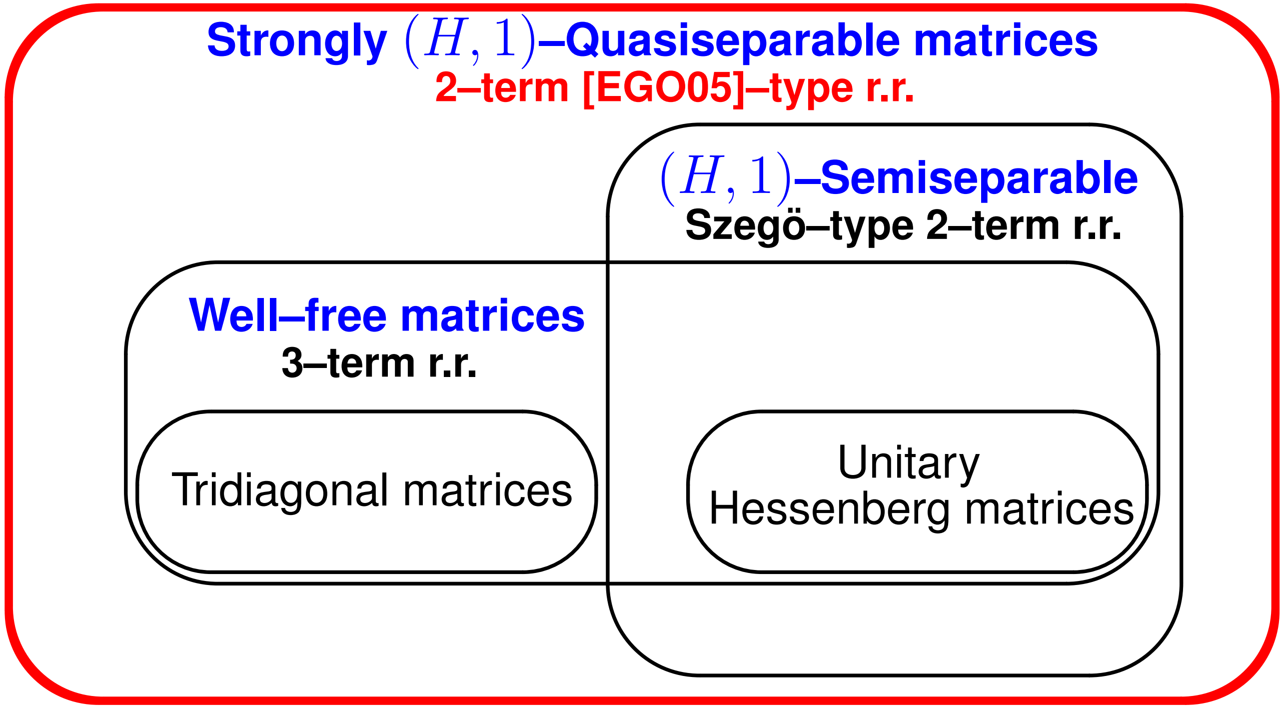
⇔
conversions

$$\{\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k\}$$

Recurrence relation coefficients

Full Characterization of $(H, 1)$–Quasiseparable Matrices

Corresponding recurrence relations



Strongly $(H, 1)$–Quasiseparable matrices

2–term [EGO05]–type r.r.

$(H, 1)$–Semiseparable

Szegö–type 2–term r.r.

Well–free matrices

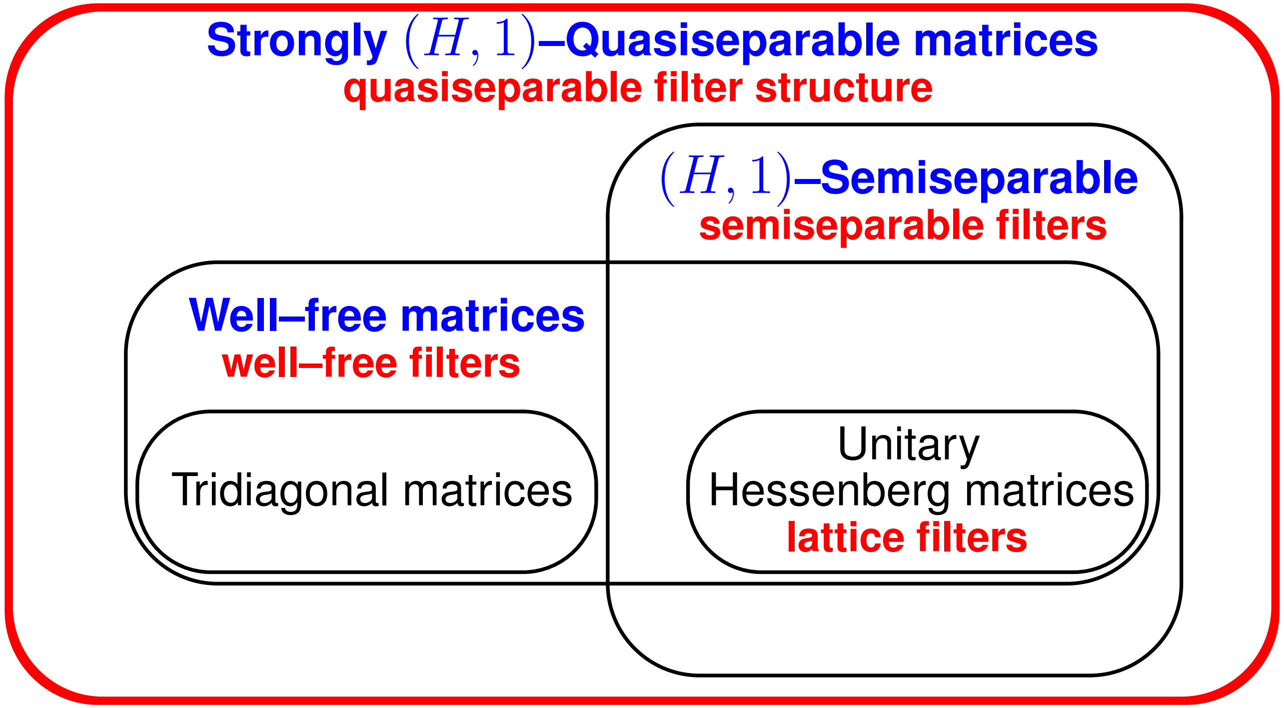
3–term r.r.

Tridiagonal matrices

Unitary
Hessenberg matrices

Full Characterization of $(H,1)$–Quasiseparable Matrices

Corresponding digital filter structures



Strongly $(H,1)$–Quasiseparable matrices
quasiseparable filter structure

$(H,1)$–Semiseparable
semiseparable filters

Well–free matrices
well–free filters

Tridiagonal matrices

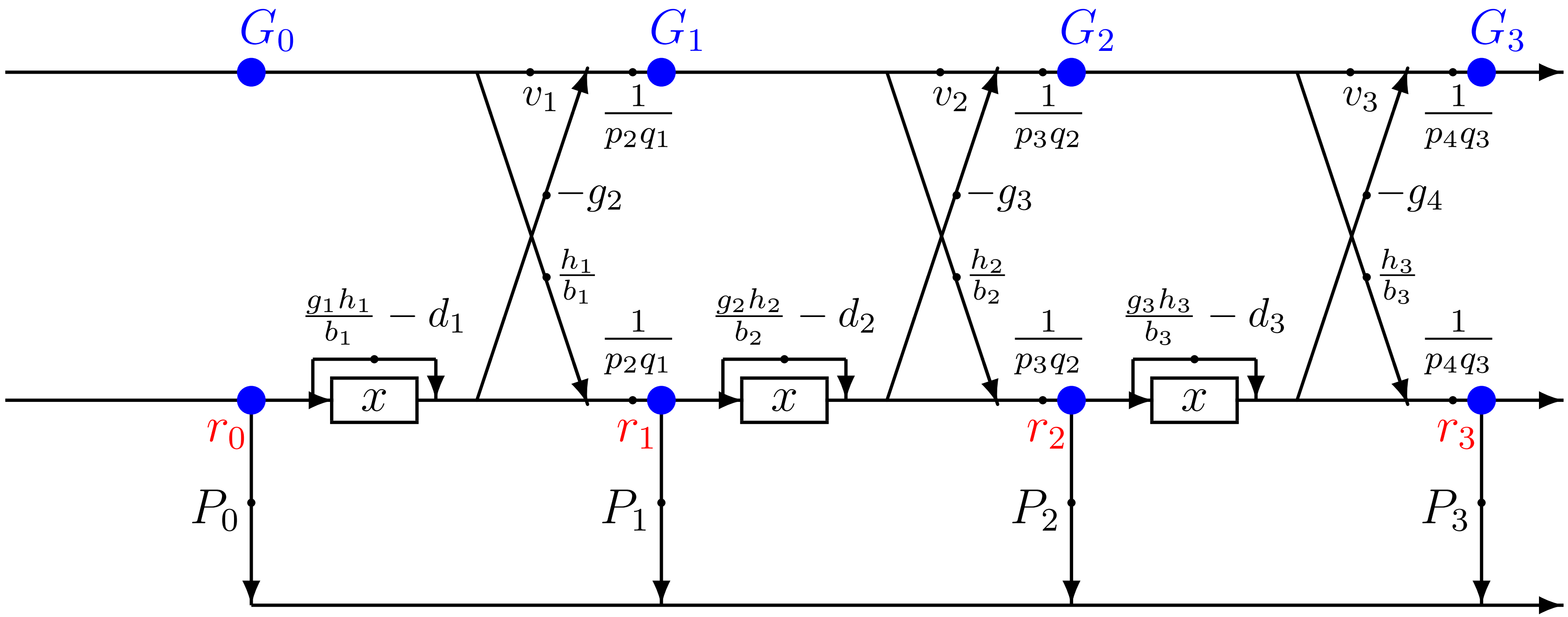
Unitary
Hessenberg matrices
lattice filters

Semiseparable filter structures

- **Theorem.** Matrix A is $(H,1)$–semiseparable if and only if the polynomials

$$r_k(x) = \det(xI - A)_{(k\times k)}$$

admit the following lattice-like realization

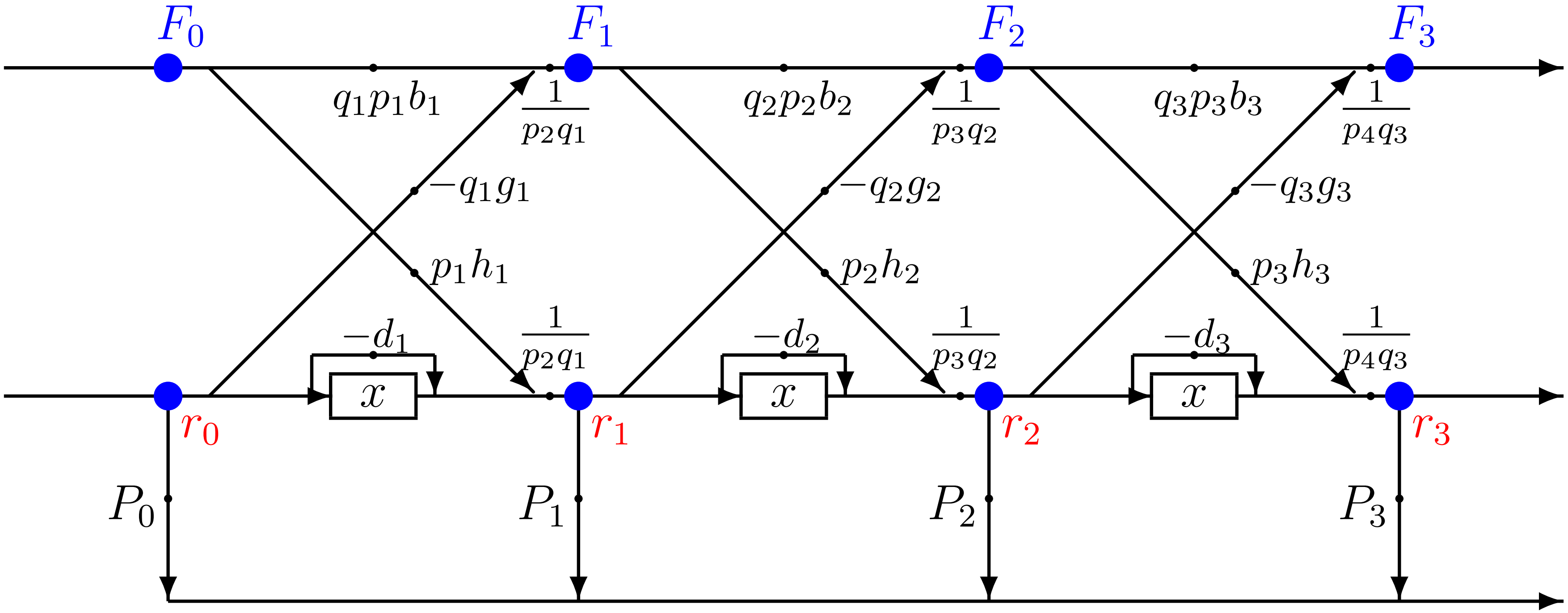


Quasiseparable filter structures

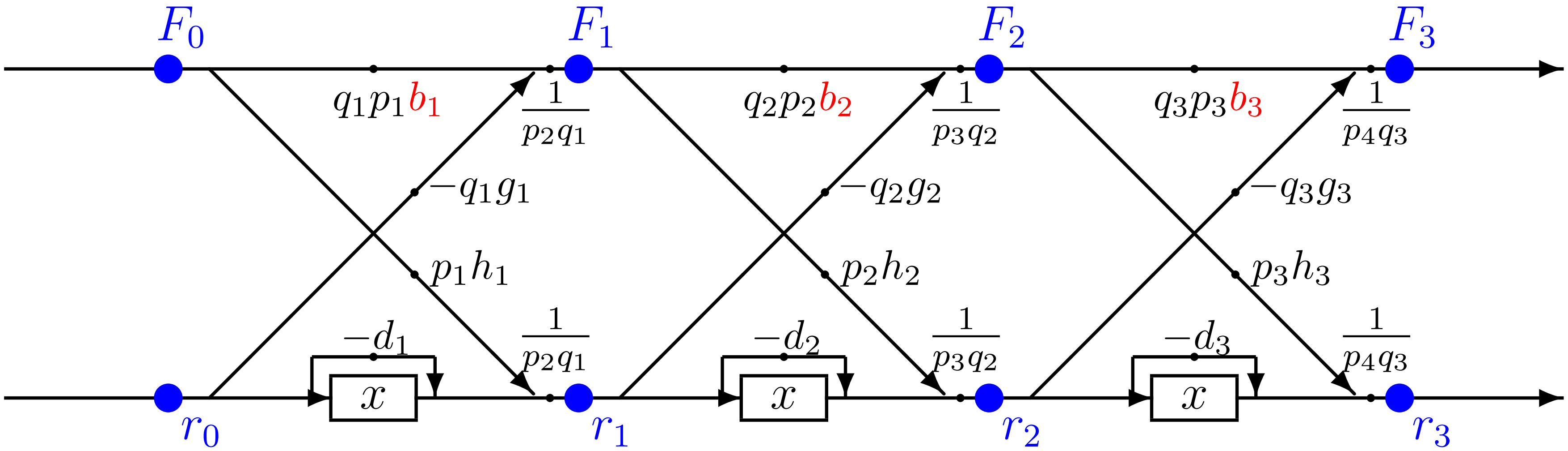
- **Theorem.** Matrix A is $(H,1)$–quasiseparable if and only if the polynomials

$$r_k(x) = \det(xI - A)_{(k\times k)}$$

admit the following lattice-like realization

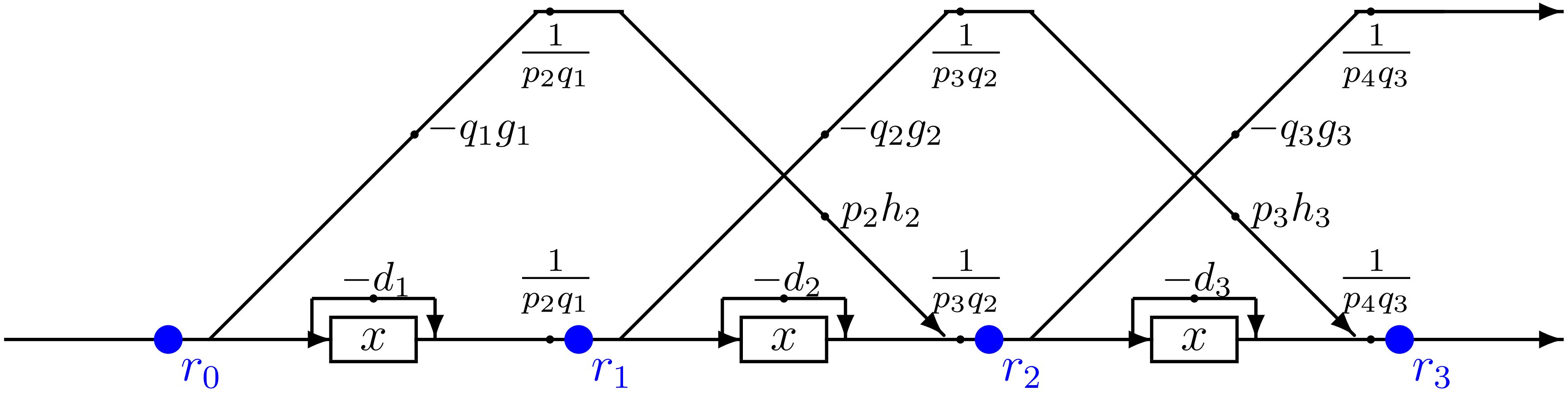


Signal flow graph for real orthogonal polynomials using quasiseparable filter structure



$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ & & p_4q_3 & d_4 & g_4h_5 \\ & & & p_5q_4 & d_5 \end{bmatrix}$$

Signal flow graph for real orthogonal polynomials using quasiseparable filter structure



$$\begin{bmatrix} d_1 & g_1h_2 & & & \\ p_2q_1 & d_2 & g_2h_3 & & \\ & p_3q_2 & d_3 & g_3h_4 & \\ & & p_4q_3 & d_4 & g_4h_5 \\ & & & p_5q_4 & d_5 \end{bmatrix}$$

Recurrence relation classification of (H, m)–quasiseparable matrices

Joint work with Vadim Olshevsky and Pavel Zhlobich

A Special Case: upper bandwidth m matrix

m nonzero superdiagonals

$$C = \begin{bmatrix} \star & \overbrace{\star & \cdots & \star} & & & & \\ \star & \star & \star & \cdots & \star & & & \\ & \star & \star & \star & \cdots & \star & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \star & \star & \star & \cdots & \star \\ & & & & \ddots & \ddots & \ddots & \vdots \\ & & & & & \star & \star & \star \\ & & & & & & \star & \star \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k\times k})$ associated with C satisfy the $(m+2)$–term recurrence relations

$$r_k(x) = \underbrace{(a_{k,k}x - a_{k-1,k})r_{k-1}(x) - a_{k-2,k}r_{k-2}(x) - \cdots - a_{k-m-1,k}r_{k-m-1}(x)}_{\text{the formula for } r_k \text{ involves the previous } m+1 \text{ polynomials}}$$

A Special Case: upper bandwidth 2 matrix

$$C = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{bmatrix}$$

A Special Case: upper bandwidth 2 matrix

$$C = \left[\begin{array}{c|ccccc}
{\color{orange}\star} & {\color{green}\star} & {\color{green}\star} & 0 & 0 & 0 \\ \hline
{\color{blue}\star} & {\color{orange}\star} & {\color{green}\star} & {\color{green}\star} & 0 & 0 \\
0 & {\color{blue}\star} & {\color{orange}\star} & {\color{green}\star} & {\color{green}\star} & 0 \\
0 & 0 & {\color{blue}\star} & {\color{orange}\star} & {\color{green}\star} & {\color{green}\star} \\
0 & 0 & 0 & {\color{blue}\star} & {\color{orange}\star} & {\color{green}\star} \\
0 & 0 & 0 & 0 & {\color{blue}\star} & {\color{orange}\star}
\end{array}\right]$$

A Special Case: upper bandwidth 2 matrix

$$C = \left[\begin{array}{cc|cccc} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \hline 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{array}\right]$$

A Special Case: upper bandwidth 2 matrix

$$C = \left[\begin{array}{ccc|ccc} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ \hline 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{array}\right]$$

A Special Case: upper bandwidth 2 matrix

$$C = \left[\begin{array}{cccc|cc} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ 0 & \star & \star & \star & \star & 0 \\ 0 & 0 & \star & \star & \star & \star \\ \hline 0 & 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 & \star & \star \end{array}\right]$$

A Special Case: upper bandwidth 2 matrix

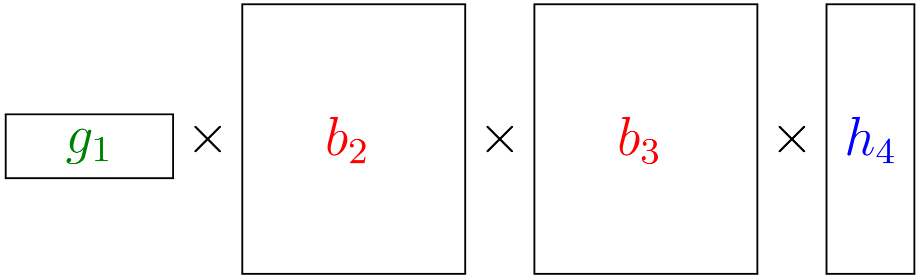
$$C = \left[\begin{array}{ccccc|c}
\star & \star & \star & 0 & 0 & 0 \\
\star & \star & \star & \star & 0 & 0 \\
0 & \star & \star & \star & \star & 0 \\
0 & 0 & \star & \star & \star & \star \\
0 & 0 & 0 & \star & \star & \star \\
\cline{1-6}
0 & 0 & 0 & 0 & \star & \star
\end{array}\right]$$

A generator representation for (H, m)–quasiseparable polynomials.

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix}$$

A generator representation for (H, m)–quasiseparable polynomials.

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \boxed{g_1b_2b_3h_4} & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix}$$

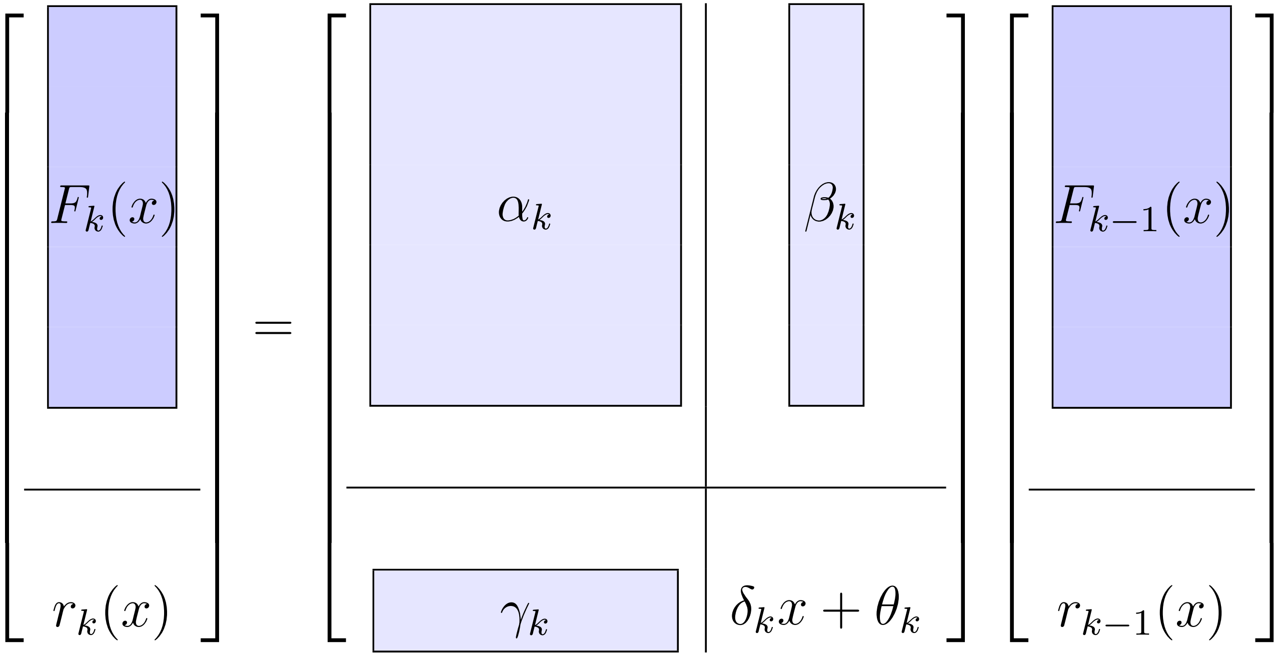
$(H, 1)$–quasiseparable generators are **scalars**	(H, m)–quasiseparable generators are **matrices**
$\boxed{g_1} \times \boxed{b_2} \times \boxed{b_3} \times \boxed{h_4}$	 $g_1 \times b_2 \times b_3 \times h_4$

What recurrence relations are satisfied by (H,m)–quasiseparable polynomials?

- The recurrence relations satisfied by $(H,1)$–quasiseparable polynomials are

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

- The recurrence relations satisfied by (H,m)–quasiseparable polynomials are



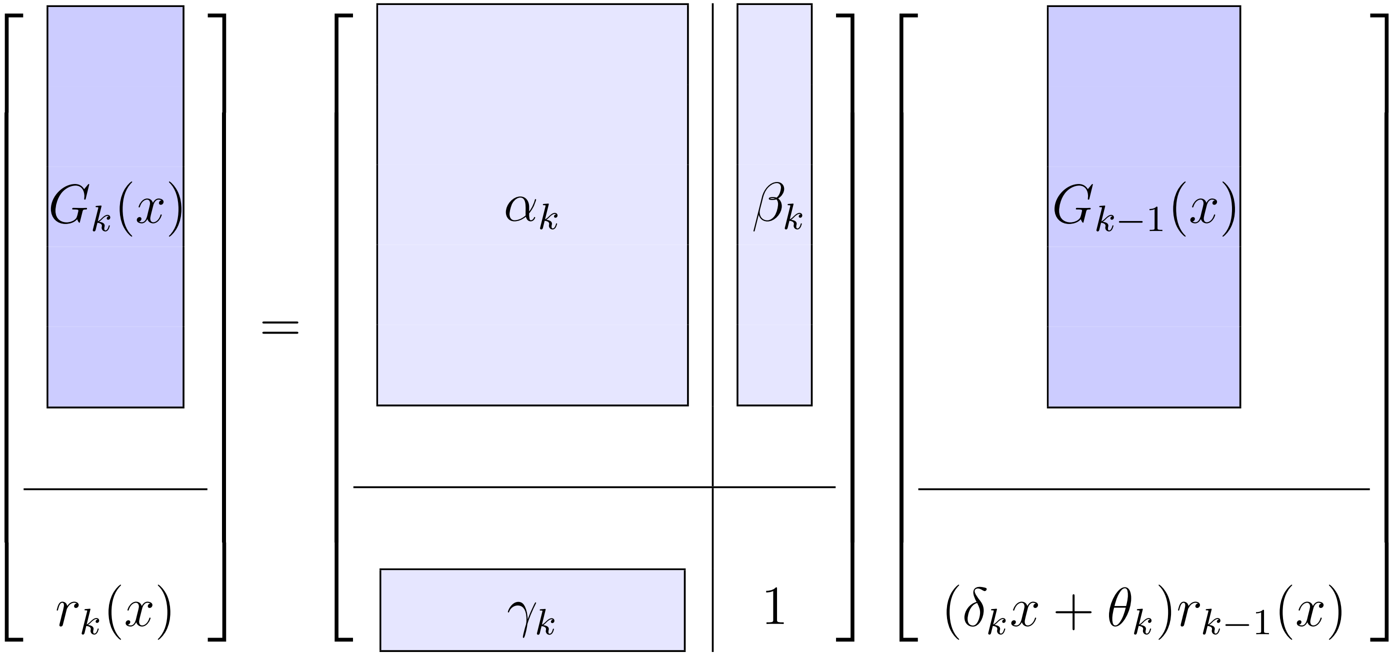
$$\left[\begin{array}{c} F_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & \delta_k x + \theta_k \end{array}\right] \left[\begin{array}{c} F_{k-1}(x) \\ \hline r_{k-1}(x) \end{array}\right]$$

What recurrence relations are satisfied by (H,m)–semiseparable polynomials?

- The recurrence relations satisfied by $(H,1)$–semiseparable polynomials are

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}$$

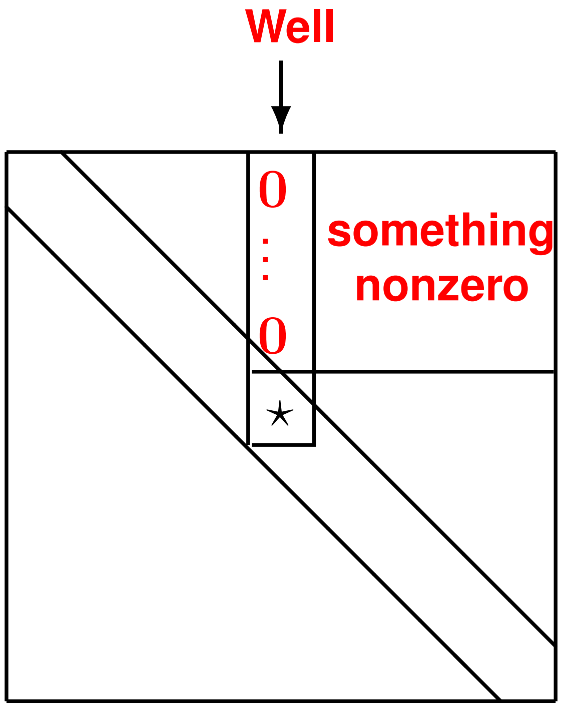
- The recurrence relations satisfied by (H,m)–semiseparable polynomials are



$$\left[\begin{array}{c} G_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & 1 \end{array}\right] \left[\begin{array}{c} G_{k-1}(x) \\ \hline (\delta_k x + \theta_k) r_{k-1}(x) \end{array}\right]$$

A generalization of well–free structure?

- Recall that a matrix is $(H, 1)$–well–free if it contains no “wells” of the form



- What is the **order** m version of this structure?

(H, m)–well–free matrices

- **Definition.** A matrix is (H, m)–well–free if adding the next column to any m consecutive columns of C_{12} does **not** increase the rank.

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- **Example.** $m = 3$.

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array}\right], \quad C_{12} = \begin{bmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \end{bmatrix}$$

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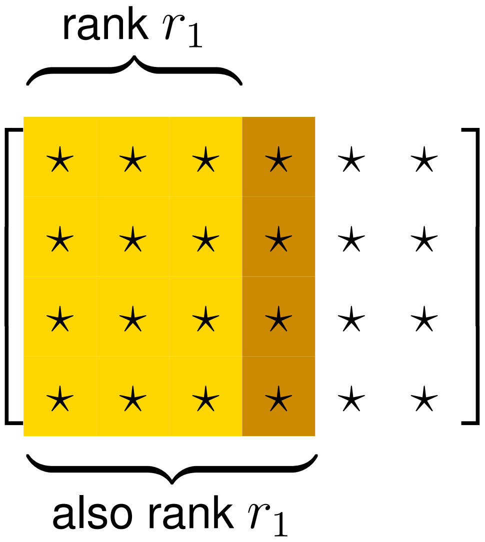

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$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array}\right], \quad C_{12} = \left[\begin{matrix} \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \end{matrix} \; \underbrace{\overbrace{\begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix}}^{\text{rank } r_3} \; \begin{matrix} \star \\ \star \\ \star \\ \star \end{matrix}}_{\text{also rank } r_3}\right]$$

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- **Example.** $m = 1$. (i.e., $(H,1)$–well–free)

$$C = \left[\begin{array}{c|c} \star & C_{12} \\ \hline \star & \star \end{array}\right], \quad C_{12} = \begin{bmatrix} 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \end{bmatrix}$$

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What recurrence relations are satisfied by (H, m)–well–free polynomials?

- The recurrence relations satisfied by $(H, 1)$–well–free polynomials are

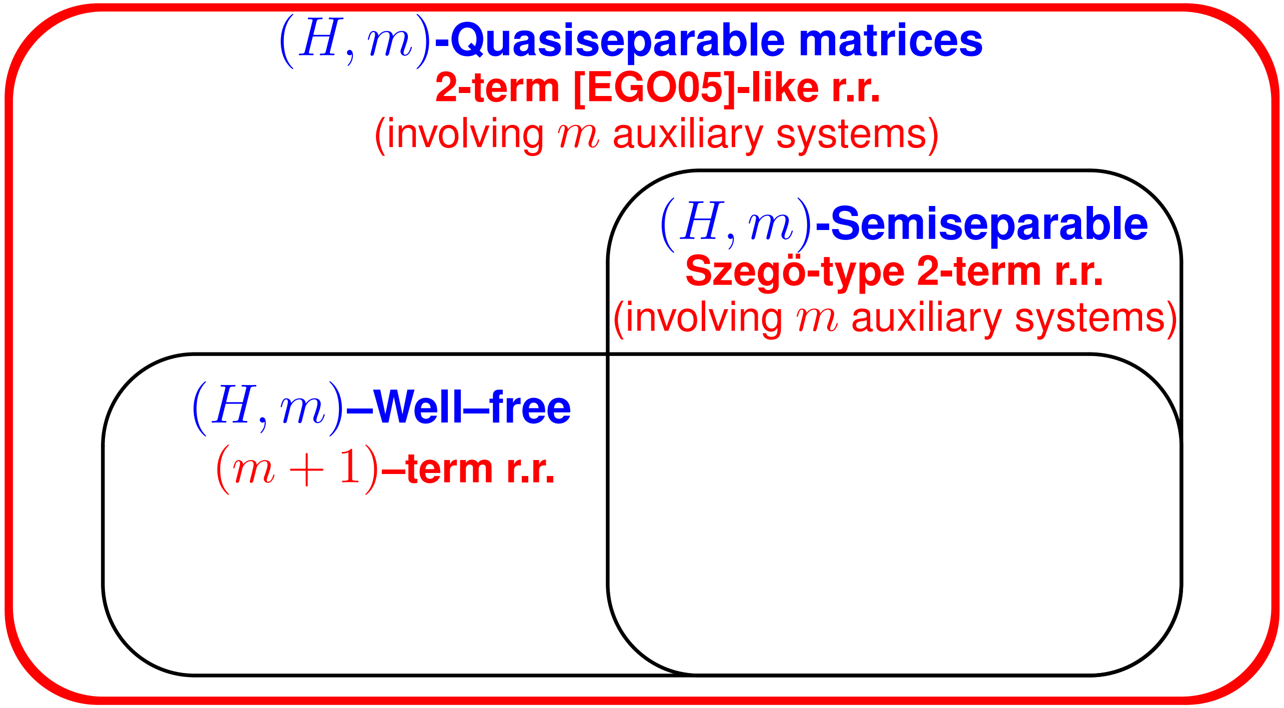
$$r_k(x) = \underbrace{(\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)}_{\text{depends on the previous two polynomials}}$$

- The recurrence relations satisfied by (H, m)–well–free polynomials are

$$r_k(x) = \underbrace{(\delta_{k,k} x + \epsilon_{k,k}) r_{k-1}(x) + (\delta_{k-1,k} x + \epsilon_{k-1,k}) r_{k-2}(x) + \cdots}_{\text{depends on the previous } m+1 \text{ polynomials}}$$

Full Characterization of (H, m)–quasiseparable matrices

Corresponding recurrence relations



(H, m)**-Quasiseparable matrices**
2-term [EGO05]-like r.r.
(involving m auxiliary systems)

(H, m)**-Semiseparable**
Szegö-type 2-term r.r.
(involving m auxiliary systems)

(H, m)**–Well–free**
$(m + 1)$**–term r.r.**

Classifications of quasiseparable matrices in terms of recurrence relations

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Joint work with Yuli Eidelman, Israel Gohberg,
Vadim Olshevsky, & Pavel Zhlobich